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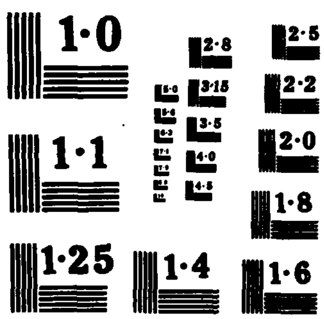
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CENTRE DE RECHERCHES

DE L'INSTITUT SUPERIEUR INDUSTRIEL CATHOLIQUE

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Stresses and Displacements in a four layered System with fixed Bottom

Contract: DAJA-85-C-0013

Intermediate Report by
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Stresses and Displacements in a four layered system with fixed bottom.

Intermediate Report.

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1

STRESSES AND DISPLACEMENTS IN A FOUR LAYERED SYSTEM WITH FIXED BOTTOM

Intermediate report.

Contract: DAJA-85-C-0013 April 30, 1985.

Introduction. This document describes

The research work necessary to fullfill the requirements of the contract that must lead to the establishment of a computer program able to calculate all stresses and displacements in a four layered system with fixed bottom submitted to a series of loads, is based on:

- existing material: isotropic multilayer theory (BURMISTER, 1943) and anisotropic multilayer theory (VAN CAUWELAERT, 1983);
- original research work: interface conditions (fixed bottom, partial friction) and satisfactory convergency at the surface and in the first layer of the system.

This intermediate report will only deal with the original research work that had to be performed. The required original research is completely terminated, which jusitifies this report, and at an entirely satisfactory level as will be shown.

This report contains three parts:

1) A complete discussion of the interface conditions.

2) The demonstration that satisfying convergency can be obtained, and in the meantime overflow problems can be eliminated, if the equations are written in closeform although in a sufficient comprehensive form so that the whole problem can still be overlooked but in such a way that all numerical problems can be solved. and

3) The complete algebraical analysis leading to the equations in extended form for an isotropic four-layered system. The analysis is also developed for a three-layered system to enable us to compare and to check the results with those obtained by means of reliable existing programs.

1. The interface conditions.

The stresses and displacements in a layer of a multilayered system are obtained for an isotropic body from following stress function (BURMISTER, 1943):

$$\phi = pa\int_0^\infty \frac{J_0(mr).J_1(ma)}{m}\left[A_i e^{mz} - B_i e^{-mz} + zC_i e^{mz} - zD_i e^{-mz}\right]dm$$

and are given by

$$\sigma_z = pa\int_0^\infty J_1(mr).J_1(ma)\Big[A_i m^2 e^{mz} + B_i m^2 e^{-mz} - C_i m(1-2\mu_i - mz)e^{mz} + D_i m(1-2\mu_i + mz)e^{-mz}\Big]dm$$

$$\sigma_r = -pa\int_0^\infty J_0(mr).J_1(ma)\Big[A_i m^2 e^{mz} + B_i m^2 e^{-mz} + C_i m(1+2\mu_i + mz)e^{mz} - D_i m(1+2\mu_i - mz)e^{-mz}\Big]dm$$
$$+ pa\int_0^\infty \frac{J_1(mr).J_1(ma)}{mr}\Big[A_i m^2 e^{mz} + B_i m^2 e^{-mz} + C_i m(1+mz)e^{mz} - D_i m(1-mz)e^{-mz}\Big]dm$$

$$\tau_{rz} = -pa\int_0^\infty J_1(mr).J_1(ma)\Big[A_i m^2 e^{mz} - B_i m^2 e^{-mz} + C_i m(2\mu_i + mz)e^{mz} + D_i m(2\mu_i - mz)e^{-mz}\Big]dm$$

$$w = \frac{1+\mu_i}{E_i}pa\int_0^\infty \frac{J_0(mr).J_1(ma)}{m}\Big[A_i m^2 e^{mz} - B_i m^2 e^{-mz} - C_i m(2-4\mu_i - mz)e^{mz} - D_i m(2-4\mu_i + mz)e^{-mz}\Big]dm$$

$$u = -\frac{1+\mu_i}{E_i}pa\int_0^\infty \frac{J_1(mr).J_1(ma)}{m}\Big[A_i m^2 e^{mz} + B_i m^2 e^{-mz} + C_i m(1+mz)e^{mz} - D_i m(1-mz)e^{-mz}\Big]dm$$

where

- a is the radius of a uniformly distributed circular load
- p is the value of the vertical pressure
- r is the horizontal distance from the axle in a cylindrical coordinate system
- z is the depth
- σ_z is the vertical stress
- σ_r is the horizontal radial stress
- τ_{rz} is the shearstress
- w is the vertical deflection
- u is the radial (horizontal) displacement
- E_i is the Young modulus of the concerned layer
- μ_i is Poisson's ratio of the concerned layer
- A_i, D_i are unknown parameters to be determined by the boundary conditions
- J_0 is the Besselfunction of the first kind of order zero
- J_1 is the Besselfunction of the first kind of order one
- m is an integrating parameter

In the case on an anisotropic body they are obtained from (VAN CAUWELAERT, 1983):

$$\phi = pa\int_0^\infty \frac{J_0(mr).\,J_1(ma)}{m}\left[A_i e^{mz} - B_i e^{-mz} + C_i e^{ms_iz} - D_i e^{-ms_iz}\right]dm$$

This stressfunction differs fundamentally from the preceeding one: indeed in putting $s_i = 1$ in it, we do not obtain the stress function for the isotropic case. We conclude that the two cases must be handled in a separate way.
The stresses and displacements are given by

$$\sigma_z = pa\int_0^\infty J_0(mr).\,J_1(ma)\Big[n_i(1+\mu_i)\left(A_i m^2 e^{mz} + B_i m^2 e^{-mz}\right) + n_i(n_i+\mu_i)\left(C_i s_i m^2 e^{s_imz} + D_i s_i m^2 e^{-s_imz}\right)\Big]dm$$

$$\sigma_r = -pa\int_0^\infty J_0(mr).\,J_1(ma)\Big[n_i(1+\mu_i)\left(A_i m^2 e^{mz} + B_i m^2 e^{-mz}\right) + \frac{n_i(n_i-\mu_i^2)}{n_i-\mu_i}\left[C_i s_i m^2 e^{s_imz} + D_i s_i m^2 e^{-s_imz}\right]\Big]dm$$

$$+\,pa\int_0^\infty \frac{J_1(mr).\,J_1(ma)}{mr}\, n_i(1+\mu_i)\Big[A_i m^2 e^{mz} + B_i m^2 e^{-mz} + C_i s_i m^2 e^{s_imz} + D_i s_i m^2 e^{-s_imz}\Big]dm$$

$$\tau_{rz} = -pa\int_0^\infty J_1(mr).\,J_1(ma)\Big[n_i(1+\mu_i)\left(A_i m^2 e^{mz} - B_i m^2 e^{-mz}\right) + s_i n_i(n_i+\mu_i)\left(C_i s_i m^2 e^{s_imz} - D_i s_i m^2 e^{-s_imz}\right)\Big]dm$$

$$w = \frac{1+\mu_i}{E_i} pa \int_0^\infty \frac{J_0(mr).J_1(ma)}{m} \Big[n_i(1+\mu_i)\left(A_i m^2 e^{mz} - B_i m^2 e^{-mz}\right) + \frac{n_i s_i (n_i+\mu_i)^2}{(1+\mu_i)} \left(C_i s_i m^2 e^{s_i m z} - D_i s_i m^2 e^{-s_i m z}\right)\Big] dm$$

$$u = \frac{(1+\mu_i) n_i (n_i+\mu_i)}{E_i} pa \int_0^\infty \frac{J_1(mr) J_1(ma)}{m} \Big[A_i m^2 e^{mz} + B_i m^2 e^{-mz} + C_i s_i m^2 e^{s_i m z} + D_i s_i m^2 e^{-s_i m z} \Big] dm$$

where

$n_i = E_{vi}/E_{hi}$ is the degree of anisotropy, the ratio between the vertical and the horizontal Young modulus of the concerned layer

μ_i is Poisson's ratio expressing a strain in the horizontal plane induced by a stress in the vertical direction

$s_i = \sqrt{\dfrac{n_i - \mu_i^2}{n_i^2 - \mu_i^2}}$ is the index of anisotropy.

1.1. The partial friction condition.

Let us consider a n-layered system, consisting in (n - 1) layers of a finite thickness built on a semi-infinite body.
For each layer exists a stress function $\phi_i(A_i\, B_i\, C_i\, D_i)$ with 4 unknown parameters: the total of unknown parameters is 4n.
Two parameters depend on the shape of the load at the surface

$$\sigma_z = f(p) \quad \text{for } r \leq a$$

$$\tau_{rz} = 0$$

At infinite depth stresses and displacaments must vanish and thus A_n and $C_n = 0$.
We remain with 4n - 4 = 4(n - 1) parameters to be determined with 4 conditions at each interface.
The hypothesis is introduced at this stage that under effect of the load, the layers remain individually fully in contact, which is expressed by imponing taht at the bottom of each layer and at the surface of next layer vertical stresses (σ_z), shearstresses (τ_{rz}) and vertical displacements (w) are identic.
The fourth interface condition depends on the relative adhesion in the horizontal plane between the considered layers.
The two extremes are

- full continuity, expressed by setting that the horizontal displacements (u) are identic;

- frictionless interface, by considering the interface as a principal plane and thus by setting the shearstresses equal zero.

Partial adhesion has been temptatively introduced by several authors, utilizing, in the same way as WESTERGAARD (1926), a relation between horizontal displacements and shearstress:

$$K\left(u_i - u_{i+1}\right) = \tau_{rz_i}$$

where u_i is the horizontal displacement at the bottom of the i-th layer and u_{i+1} that at the surface of the (i + 1)-th layer.
We shall prove that such a relation cannot be correct in the case of a multilayer.

One has, for an isotropic body, following relations between displacements, shearstrains and shearstresses:

$$\frac{\partial u_i}{\partial z} + \frac{\partial w}{\partial r} = \gamma_{rz_i} = \left[2(1+\mu_i)/E_i\right] \cdot \tau_{rz_i}$$

$$\frac{\partial u_{i+1}}{\partial z} + \frac{\partial w_{i+1}}{\partial r} = \gamma_{rz_{i+1}} = \left[2(1+\mu_{i+1})/E_{i+1}\right] \cdot \tau_{rz_{i+1}}$$

We know from the boundary conditions that

$$w_i = w_{i+1} \qquad \tau_{rz_i} = \tau_{rz_{i+1}}$$

This is true everywhere on the interface so that we also can write that

$$\frac{\partial w_i}{\partial r} = \frac{\partial w_{i+1}}{\partial r}$$

By substraction we obtain

$$\frac{\partial u_i}{\partial z} - \frac{\partial u_{i+1}}{\partial z} = 2\left[\frac{1+\mu_i}{E_i} - \frac{1+\mu_{i+1}}{E_{i+1}}\right]\tau_{rz_i} = k'\tau_{rz_i}$$

We can thus write

$$\frac{1}{k'}\left[\frac{\partial u_i}{\partial z} - \frac{\partial u_{i+1}}{\partial z}\right] = k\,(u_i - u_{i+1})$$

This relation must be satisfied for all values of the parameter m, so that one must necessarily have that

$$\frac{k}{k'}\frac{\partial u_i}{\partial z} = u_i \qquad \frac{k}{k'}\frac{\partial u_{i+1}}{\partial z} = u_{i+1}$$

The solutions of those differential equations are

$$u_i = e^{zk'/k} \cdot f_1(r) \qquad u_{i+1} = e^{zk'/k} \cdot f_2(r)$$

Comparing those solutions with the relation above for the horizontal displacements we conclude that the obtained expressions must be deduced from a stressfunction different from the original one which is nevertheless the unique solution of the

compatibility equations. Comp atibility is thus not respected and relation $k(u_i - u_{i+1}) = \tau_{rz}$ cannot be accepted.

Our meaning is that the only way to express partial continuity (or partial adhesion) consists in writing

$$u_i = k.u_{i+1}$$

with $k \in [0, \infty]$.
When k = 1, one has full continuity
When k $\neq$ 1, one has partial continuity.

It is necessery now to give a physical sence to the parameter k.
Excepted the extreme case of a frictionless interface, there will always be some friction between the layers at their interface.
We rely then our approach on Coulomb's definition of friction. If φ is the angle of friction at the interface, there will be no sliding (in the geotechnical sense of the word) between the two layers as long as $\tau_{rz} < \sigma_z \, tg\,\varphi$.
The limit value for k, at which sliding due to shearfailure will occur, is then given by $\tau_{rz}/\sigma_z = tg\,\varphi$.
Beyond this value of k, shearstresses vanish and the interface has to be considered as frictionless.
We must of course take into account here the stresses due to the wheel load but also those due to the own weight of the layers above the interface, which reduce to vertical stresses only.
The final relation becomes then

$$\frac{\tau_{rz}\,(\text{traffic})}{\sigma_z(\text{traffic}) + \sigma_z(\text{mass})} = tg\,\varphi.$$

The values of the stresses due to traffic varie with the distance to the axle of the load.
We carry then the calculations out in two steps:

- we calculate for different values of k the maximum value (function from the distance from the axle of the load) of the ratio τ/σ.
- we determine the limit value of k in function of $tg\varphi$.

As an illustration of the method, we have taken the most simple case, that of an isotropic two-layer system, with H the thickness of the first layer.
For simplicity we take $\mu_1 = \mu_2 = 0.5$

We write $\quad A_i m^2 = A_i \;,\; B_i m^2 = B_i \;,\; C_i m = C_i \;,\; D_i m = D_i$

$$F = \frac{E_1(1+\mu_2)}{E_2(1+\mu_1)} = \frac{E_1}{E_2}$$

$$L = k \cdot \frac{E_1(1+\mu_2)}{E_2(1+\mu_1)} = k\,\frac{E_1}{E_2}$$

The boundary conditions are then:

At the surface (z = 0):

$$\sigma_z = p \qquad A_1 + B_1 = 1$$

$$\tau_{rz} = 0 \qquad A_1 - B_1 + C_1 + D_1 = 0$$

At the interface (z = H):

$$\sigma_{z_1} = \sigma_{z_2} \qquad A_1 e^{mH} + B_1 e^{-mH} + mH\,C_1 e^{mH} + mH\,D_1 e^{-mH} = B_2 e^{-mH} + mH\,D_2 e^{-mH}$$

$$\tau_{rz_1} = \tau_{rz_2} \qquad A_1 e^{mH} - B_1 e^{-mH} + (1+mH)C_1 e^{mH} + (1-mH)D_1 e^{-mH} = -B_2 e^{-mH} + (1-mH)D_2 e^{-mH}$$

$$w_1 = w_2 \qquad A_1 e^{mH} - B_1 e^{-mH} + mH\,C_1 e^{mH} - mH\,D_1 e^{-mH} = F\left[-B_2 e^{-mH} - mH\,D_2 e^{-mH}\right]$$

$$u_1 = k u_2 \qquad A_1 e^{mH} + B_1 e^{-mH} + (1+mH)C_1 e^{mH} - (1-mH)D_1 e^{-mH} = L\left[B_2 e^{-mH} - (1-mH)D_2 e^{-mH}\right]$$

Solving the system of 6 equations, one obtains at the interface (z = H):

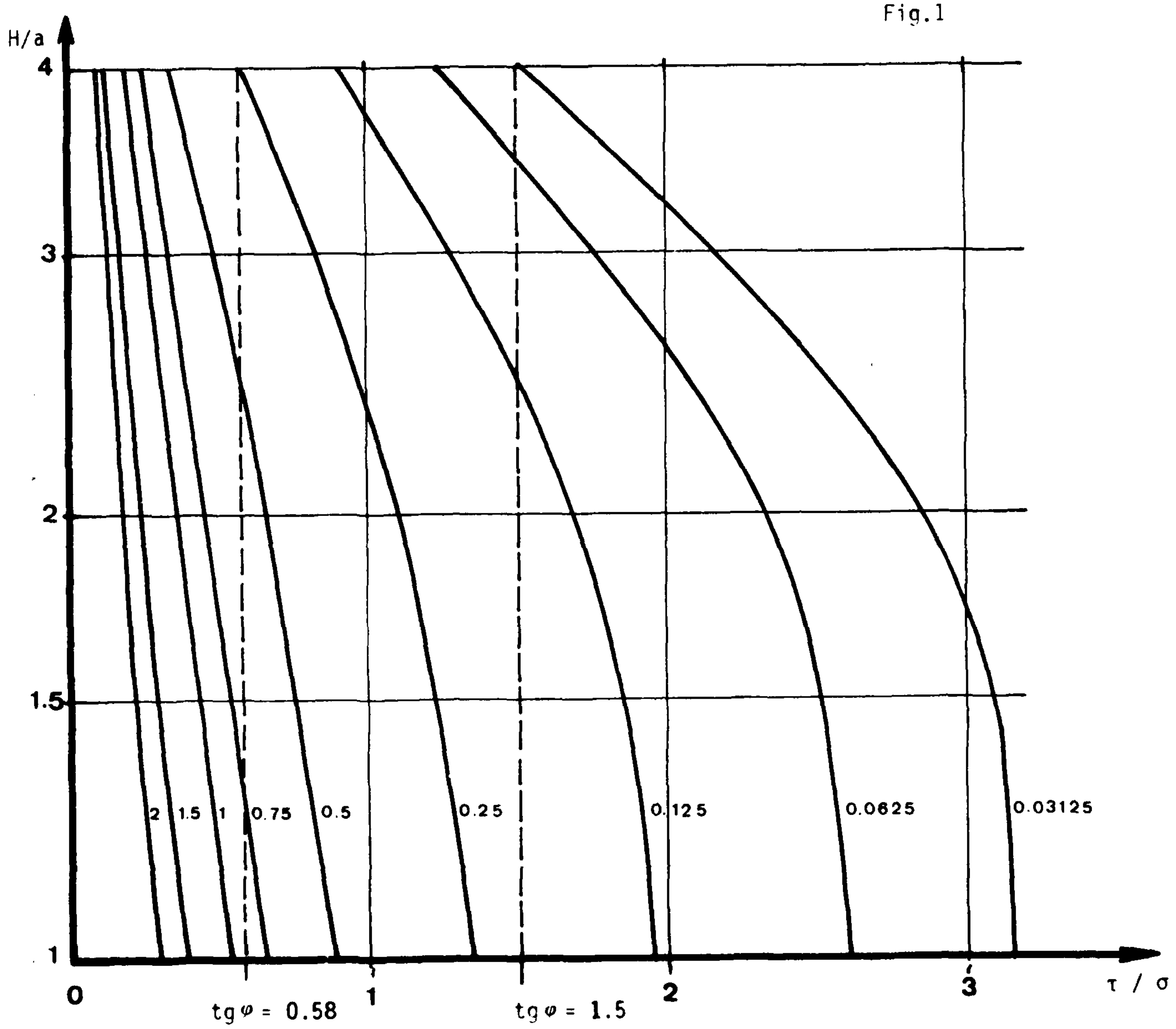
$$\sigma_z = 2pa \int_0^\infty J_0(mr)\,J_1(ma)\left[(1-mH)(1-L)e^{-3mH} + (1+mH)(1+L)e^{-mH}\right]\frac{dm}{\nabla}$$

$$\tau_{rz} = 2pa \int_0^\infty J_1(mr)\,J_1(ma)\left[(1-F)e^{-3mH} + (1+F)e^{-mH}\right]\frac{dm}{\nabla}$$

$$\nabla = (1-F)(1-L)e^{-4mH} + 2\left[(1-FL) - 2mH(F-L) + 2m^2H^2(1-FL)\right]e^{-2mH} + (1+F)(1+L)$$

We have performed the computations for different values of H/a, with a = 10 cm, p = 0,6 MN/m^2 and E_1/E_2 = 10. We have considered a specific weight of the first layer of 22 kN/m^3.
The results ar given on figure 1: in abscissa one finds the values of the ratio τ/σ and in ordinate the relative thicknesses.
The curves give for different values of k the maximum value of τ/σ.

Fig.1



This is practically impossible.
One could of course be tempted to interrupt the integration procedure when the first integral has converged to a satisfactory level, "hoping" that the second integral can be neglected at that moment.
To illustrate the danger of such an approach we return to the semi-infinite body.

In the case on an isotropic body, the deflection at the surface and the vertical stress at a depth z are given in the axle of the load by

$$w = -pa\,\frac{2(1-\mu^2)}{E}\int_0^\infty \frac{J_1(ma)}{m}\,dm \qquad (2)$$

$$\sigma_z = pa\int_0^\infty J_1(ma)(1-mz)\,e^{-mz}\,dm \qquad (3)$$

Those integrals can of course be solved analytically

$$w = -pa\,\frac{2(1-\mu^2)}{E} \qquad (4)$$

$$\sigma_z = p\left[1 - \frac{z^3}{(a^2+z^2)^{3/2}}\right] \qquad (5)$$

We can compare (2) with the second integral of (1) and (3) with the first integral of (1).
We perform then a numerical integration of (2) and (3) and stop the procedure when (3) has converged to a satisfactory level, which is easily checked by comparing the obtained result with the correct one given by (5).
The difference between the numerical result for w obtained at that moment by integration of (2) with the analytical result given by (4) will give us an illustration of the possible error when integrating (1) and stopping the process when its first integral has converged.
This difference is illustrated in figure 2.
In abciss, we have the convergency level adopted for the vertical stress and in ordinate the error, expressed in %, on the values of the vertical stress and the vertical deflection.
One sees that for even such low levels as 10^{-3}, the value of the vertical stress is absolutely correct, while the error on the deflection varies between - 5% and + 8%, depending on thechosen convergency level and the relative depth at which the vertical stress is computed; worst of all is that we have no means to predict either the direction either the amplitude of the error on w.

2.3. The vertical deflection at the surface.

When one computes the deflections at the surface, convergency is obtained only very slowly.
To illustrate this let us look at the expression of the deflection at the surface developed in § 2.1.

$$w = -pa\frac{2(1-\mu_1^2)}{E_1}\int_0^\infty \frac{J_0(mr).J_1(ma)}{m}.$$

$$\left[\frac{Fe^{2mH}-(2F-1-2mH)-(1-F)e^{-2mH}}{Fe^{2mH}+(2F-1)2mH-(1+2m^2H^2)+(1-F)e^{-2mH}}\right]dm$$

To avoid overflow problems we divide numerator and denominator by e^{2mH}

$$w = -pa\frac{2(1-\mu_1^2)}{E_1}\int_0^\infty \frac{J_0(mr).J_1(ma)}{m}.$$

$$\left\{\frac{F-(2F-1-2mH)e^{-2mH}-(1-F)e^{-4mH}}{F+[(2F-1)2mH-(1+2m^2H^2)]e^{-2mH}+(1-F)e^{-4mH}}\right\}dm$$

For large values of m, numerator and denominator tend both to F, so that for, let us say $m = m_L$, the expression above could be written as follows:

$$w = -pa\frac{2(1-\mu_1^2)}{E_1}\int_0^{m_L} \frac{J_0(mr).J_1(ma)}{m}.$$

$$\left\{\frac{F-(2F-1-2mH)e^{-2mH}-(1-F)e^{-4mH}}{F+[(2F-1)2mH-(1+2m^2H^2)]e^{-2mH}+(1-F)e^{-4mH}}\right\}dm$$

$$-pa\frac{2(1-\mu_1^2)}{E_1}\int_{m_L}^\infty \frac{J_0(mr).J_1(ma)}{m}dm \qquad (1)$$

The first integral converges fast, the second converges proportionnaly to 1/m. This means that if one should want a result correct at 10^{-5}, one has to perform the numerical integration of the second integral until values of m above 100,000!

Underflow will obviously occur now, but most of the computers have a routine that sets variables subjected to underflow equal to zero. If such a routine does not exist, it is very easy to build it into the program.
But more interesting is the fact that, having transformed the relations for C_1 and D_1, convergency will occure quite quickly and in a complete safe way: the numerators both tend to zero, while the denominator tends to a constant F.

This can be obtained automatically in writing the boundary counditions at the surface (z = -H) as follows:

$$A_1 e^{-3mH} + B_1 e^{-mH} - C_1(1-2\mu_1+mH)e^{-3mH} + D_1(1-2\mu_1-mH)e^{-mH} = e^{-2mH}$$
$$A_1 e^{-3mH} - B_1 e^{-mH} + C_1(2\mu_1-mH)e^{-3mH} + D_1(2\mu_1+mH)e^{-mH} = 0$$

However this is only true in the case of a two-layer system.
In a three-layer with H_1, the thickness of the first layer, and H_2, the thickness of the second layer, occur exponents such as

$$e^{2mH_1 \cdot 2mH_2} \quad \text{and} \quad e^{-2mH_1 \cdot 2mH_2}$$

But they eliminate when writing the denominator in closeform so that dividing the expressions by the largest out of e^{2mH_1} and e^{2mH_2} is enough. If one should divide by $e^{2mH_1 \cdot 2mH_2}$, the denominator would also tend to zero, which should stop the program because of dividing by zero.

Here is another reason for writing the equations in closeform for three- and more-layers.

2.2. Over- and underflow problems.

During the integration procedure m varies from 0 to a value high enough to ensure convergency. We mean by this that the integration procedure can be stopped from the moment on that the terms of the series become so small that they have no more influence on the final result and can thus be neglected.

Practically, however this means that m can reach quite high values such as 20 or 30 for example.

To illustrate the influence of this, let us go back to the two-layer developed in the preceeding paragraph.
The values of C_1 and D_1, from which the values of all the other parameters can be deduced, are

$$C_1 = \frac{\left[(1-F+mH)e^{mH} - (1-F)e^{-mH}\right]}{Fe^{2mH} + (2F-1).2mH - (1+2m^2H^2) + (1-F)e^{-2mH}}$$

$$D_1 = \frac{\left[Fe^{mH} - (F-mH)e^{-mH}\right]}{Fe^{2mH} + (2F-1).2mH - (1+2m^2H^2) + (1-F)e^{-2mH}}$$

The geometrical unities are generally expressed in function of a, the radius of the load.
Let us consider H/a = 5.
One immediately sees that no computer can handle exponents as $e^{mH/a}$ and $e^{2mH/a}$ without overflow occuring for values of m above 10.
However this problem can easily be solved by dividing both numerator and denominator by e^{2mH}:

$$C_1 = \frac{\left[(1-F+mH)e^{-mH} - (1-F)e^{-3mH}\right]}{F + \left[(2F-1).2mH - (1+2m^2H^2)\right]e^{-2mH} + (1-F)e^{-4mH}}$$

$$D_1 = \frac{\left[Fe^{-mH} - (F-mH)e^{-3mH}\right]}{F + \left[(2F-1).2mH - (1+2m^2H^2)\right]e^{-2mH} + (1-F)e^{-4mH}}$$

Replacing A_1, B_1, C_1 and D_1 by their values, the deflection becomes

$$w = -pa\,\frac{2(1-\mu_1^2)}{E_1}\int_0^\infty \frac{J_0(mr)\,J_1(ma)}{m}\,.$$

$$\left[\frac{Fe^{2mH} - (2F-1-2mH) - (1-F)\,e^{-2mH}}{Fe^{2mH} + (2F-1)2mH - (1+2m^2H^2) + (1-F)\,e^{-2mH}}\right].\,dm$$

At the origin of the integration (m = 0), the term in between brackets becomes indefinite: 0/0.

This has no influence when computing stresses, because the Bessel functions products occuring here are also zero at the origin: $J_0(mr).J_1(ma) = 0$ for m = 0.

But in the case of the deflection

$$\lim_{m\to 0}\frac{J_0(mr).J_1(ma)}{m} = \frac{a}{2}$$

It is therefore absolutely necessary to have the term in between brackets in close form to be able to determine its value for m = 0..

The importance of the first term of the series is not negligible: for m = 0 the term in between brackets is equal to $E_1.(1-\mu_2^2)\,/\,[E_2.(1-\mu_1^2)]$.

If h is the interval choosen for the numerical integration, one can then write

$$w = -pa\,\frac{2(1-\mu_1^2)}{E_1}\left\{\frac{1}{2}\cdot\frac{E_1(1-\mu_2^2)}{E_2(1-\mu_1^2)}\cdot\frac{h}{3} + \int_h^\infty \frac{J_0(mr)J_1(ma)}{m}[\quad]\,dm\right\}$$

and, if we make a semi-infinite body from the two-layer ($E_1 = E_2$, $\mu_1 = \mu_2$):

$$w = -pa\,\frac{2(1-\mu^2)}{E}\left\{\frac{h}{6} + \int_h^\infty \frac{J_0(mr).J_1(ma)}{m}[\quad]\,dm\right\}$$

Comparing this expression with that for the deflection at the surface of a semi-infinite body

$$w_\infty = -pa.2(1-\mu^2)/E$$

one concludes that the contribution of the first term h/6 is indeed not negligible, especially when we have in mind that the only practical measurement that can be performed on a real roadstructure is the vertical deflection at the surface.

At the surface (z = -H):

$$A_1 e^{-mH} + B_1 e^{mH} - C_1(1-2\mu_1+mH)e^{-mH} + D_1(1-2\mu_1-mH)e^{mH} = 1$$

$$A_1 e^{-mH} - B_1 e^{mH} + C_1(2\mu_1-mH)e^{-mH} + D_1(2\mu_1+mH)e^{mH} = 0$$

At the frictionless interface (z = 0):

$$A_1 + B_1 - C_1(1-2\mu_1) + D_1(1-2\mu_1) = B_2 + D_2(1-2\mu_2)$$

$$A_1 - B_1 + 2\mu_1 C_1 + 2\mu_1 D_1 = 0$$

$$-B_2 + 2\mu_2 D_2 = 0$$

$$\frac{1+\mu_1}{E_1}\left[A_1 - B_1 - 2C_1(1-2\mu_1) - 2D_1(1-2\mu_1)\right]$$

$$= \frac{1+\mu_2}{E_2}\left[-B_2 - 2D_2(1-2\mu_2)\right]$$

Solving the system for C_1 and D_1, one obtains (BURMISTER, 1943)

$$C_1 = \left[(1-F+mH)e^{mH} - (1-F)e^{-mH}\right]\cdot\frac{1}{\nabla}$$

$$D_1 = \left[Fe^{mH} - (F-mH)e^{-mH}\right]\cdot\frac{1}{\nabla}$$

where

$$\nabla = Fe^{2mH} + (2F-1)\cdot 2mH - (1+2m^2H^2) + (1-F)e^{-2mH}$$

$$F = \frac{(1-\mu_2)+n(1-\mu_1)}{2(1-\mu_2)} \qquad n = \frac{E_2}{E_1}\cdot\frac{(1+\mu_1)}{(1+\mu_2)}$$

$$A_1 = C_1(F-2\mu_1) - D_1(1-F)$$

$$B_1 = C_1F - D_1(1-2\mu_1-F)$$

The vertical deflection at the surface is

$$w = pa\frac{1+\mu_1}{E_1}\int_0^\infty \frac{J_0(mr)\,J_1(ma)}{m}\left[A_1m^2e^{-mH} - B_1m^2e^{mH}\right.$$

$$\left. - (2-4\mu_1+mH)C_1me^{-mH} - (2-4\mu_1-mH)D_1me^{mH}\right]dm$$

2. Solution of particular numerical problems (convergency-problems).

2.1 The full slip interface condition.

The value of any stress or displacement is obtained from one of the above mentioned relations.
Let us consider, for example, the vertical stress in the i-th layer of an anisotropic layer:

$$\sigma_z = pa \int_0^\infty J_0(mr) \cdot J_1(ma) \left[n_i(1+\mu_i)\left(A_i m^2 e^{mz} + B_i m^2 e^{-mz}\right) + n_i(n_i+\mu_i)\left(C_i s_i m^2 e^{s_i m z} + D_i s_i m^2 e^{-s_i m z}\right)\right] dm$$

The integration can only be performed numerically.
Thus one must calculate the value of the stress for a set of values of m growing from 0 to a value high enough to ensure convergency.
For each value of m, those of the parameters A_i, B_i, C_i and D_i must be determinated out of the set of boundary counditions, a system of (4n - 1) equations with (4n - 1) unknowns in the case of a fixed bottom and n layers above it.
The first programs solved this problem by inverting the matrix of the (4n - 1) unknowns. Nevertheless the inversion procedure leads in some cases to unsoluble difficulties because of the presence of the negative exponents tending to zero in the determinant of the denominator.
Other programs have tried to avoid the inversionprocedure as follows: one chooses appropriate values for B_n and D_n, goes trough the whole set of equations and verifies in how far the surface conditions are met. One then chooses another pair of values for B_n and D_n and follows the same procedure. Since the whole process is linear, the correct values for B_n and D_n can finally be obtained by linear interpolation after two runs. The difficulty lies in the appropriate choice of the values of B_n and D_n to ensure a numerically correct interpolation.
However, even those programs are not entirely appropriate for the cases with frictionless conditions at some interfaces.
We shall show this with the most simple case, that of a two layer.
Writing A_i, B_i, C_i, D_i instead of $A_i m^2, B_i m^2, C_i m, D_i m$, the boundary conditions are in the case of two isotropic layers, with the origin (z = 0) at the interface, the thickness of the first layer being H and the second layer semi-infinite:

We also conclude that the relative influence on the deflection is much more important when we fix the horizontal displacements ($u = 0$). This should be the case in a laboratory testpit with lateral walls, but less in the case of a real road where lateral movements are not restricted.

The relative influence of the condition $\tau_{rz} = 0$ is also more important than that of the condition $w = 0$, altough less important than the condition $u = 0$. It seems nevertheless very unlikely that there would be no friction between the subground and the last layer.

The easiest way to fix the bottom from a mathematical point of view is on the other hand the condition $w = 0$.

Taking then into account the little influence of the chosen condition on the deflection at the surface and the fact that conditions $u = 0$ and $\tau_{rz} = 0$ have less physical sense, we shall retain the condition $w = 0$ as the most indicated fixed bottom condition.

The deflection at the surface is given by the same relation as above with the appropriate value for C.

One sees that in the 3 cases, the deflection is composed of a first term

$$w_\infty = pa\,\frac{s(1-n)}{E(1-s)}\int_0^\infty \frac{J_1(am)}{m}\,dm = pa\,\frac{s(1-n)}{E(1-s)}$$

This term is the deflection on top of a semi-infinite body.

In the case of an isotropic body, one has (n = 1) :

$$w_\infty = -pa\,\frac{2(1-\mu^2)}{E}$$

The second term w_r depends on the choosen boundary condition; but in the three cases it reduces the value of w_∞ because of the fixed bottom.

We have computed the values of w_∞ and w_r for different values of H/a. The results are given below, at a factor $(1+\mu)/E$, for s = 0.5 and for the isotropic case.

s = 0.5 $\quad w_\infty = 1.025$

H/a	$w_r\ (\tau_{rz} = 0)$	$w_r\ (u = 0)$	$w_r\ (w = 0)$
1	0.339	0.729	0.153
2	0.120	0.295	0.043
3	0.058	0.147	0.019
4	0.034	0.086	0.011
5	0.022	0.057	0.007
6	0.015	0.040	0.005
7	0.011	0.029	0.004
8	0.009	0.023	0.003
9	0.007	0.018	0.002
10	0.006	0.015	0.002

n = 1 $\quad w_\infty = 1.000$

H/a	$w_r\ (\tau_{rz} = 0)$	$w_r\ (u = 0)$	$w_r\ (w = 0)$
1	0.402	0.934	0.276
2	0.143	0.442	0.086
3	0.069	0.231	0.040
4	0.040	0.138	0.023
5	0.026	0.091	0.015
6	0.018	0.064	0.010
7	0.013	0.048	0.008
8	0.010	0.037	0.006
9	0.008	0.029	0.005
10	0.007	0.024	0.004

We conclude that from a depth of about H/a = 5, the absolute influence of the fixed boundary is negligible. This influence will still be much lesser in the case of a roadstructure where the E-moduli of the layers are sensitively higher than the modulus of the subground.

We now choose an appropriate function $f(r)$ so that w becomes zero for z = H.

$$f(r) = -\frac{1+\mu}{E}\cdot pa\int_0^{\infty}\frac{J_0(mr)\,J_1(ma)}{m}\Big[n(1+\mu)\left(Am^2e^{mH} - Bm^2e^{-mH}\right)$$

$$+\frac{ns(n+\mu)^2}{(1+\mu)}\left(Csm^2e^{smH} - Dsm^2e^{-smH}\right)\Big]dm$$

The final expression for the deflection is then

$$w = w_1 + f(r)$$

One verifies that $f(r)$ is indeed only a function of r and that w = 0 for z = H.
For the same reasons as those developed in § 1.1.2, one of the parameters A or C must be zero.
The other paramter is obtained by a supplementary boundary condition (a mechanical condition):

- $\tau_{rz} = 0$ at the depth H
- $u = 0$ at the depth H.

If we still suppose $s<1$ and thus A = 0, one obtains in the case that $\tau_{rz} = 0$

$$ns(n+\mu)Cs\,m^2 = \frac{e^{-Hm}\cdot e^{-sHm} - e^{-2sHm}}{2e^{-Hm}\cdot e^{-sHm} - (1-s) - (1+s)e^{-2sHm}}$$

The deflection at the surface is then

$$w = pa\,\frac{s(1-n)}{E(1-s)}\int_0^{\infty}\frac{J_1(am)}{m}\,dm$$

$$+\,pa\,\frac{1+\mu}{E}\int_0^{\infty}\frac{J_1(am)}{m}\Bigg\{\frac{s}{(1-s)}\,\frac{(n+\mu)}{(1+\mu)}\,e^{-sHm} - \frac{s}{1-s}\,e^{-mH}$$

$$+\,ns(n+\mu)\,Csm^2\Bigg[\frac{2s(m-1)}{(1-s)(1+\mu)} + \frac{2s}{1-s}\,e^{-mH}$$

$$-\frac{m+\mu}{1+\mu}\,s\,e^{sHm} - \frac{m+\mu}{1+\mu}\,s\,\frac{1+s}{1-s}\,e^{-smH}\Bigg]\Bigg\}\,dm$$

In the case u = 0:

$$ns(n+\mu)\,Csm^2 = \frac{e^{-sHm}\left[(n+\mu)\,s\,e^{-Hm} - (1+\mu)e^{-sHm}\right]}{2s(n+\mu)e^{-Hm}e^{-sHm} + (1+\mu)(1-s) - (1+\mu)(1+s)\,e^{-2sHm}}$$

1.2.2. Fixed bottom expressed by mechanical condition only.

Referring to the stress and displacements equations given in paragraph 1, the condition w = 0 at a depth H is written

$$n(1+\mu)(Am^2e^{mH} - Bm^2e^{-mH}) + \frac{ns(n+\mu)^2}{(1+\mu)}(Csm^2e^{smH} - Dsm^2e^{-smH}) = 0$$

Replacing B and D by their values

$$Am^2\left[n(1+\mu)e^{mH} - \frac{n(1+s)(1+\mu)}{(1-s)}e^{-mH} + \frac{2ns(n+\mu)}{(1-s)}e^{-smH}\right]$$

$$+ Csm^2\left[\frac{ns(n+\mu)^2}{(1+\mu)}e^{smH} - \frac{2ns(n+\mu)}{(1-s)}e^{-mH} + \frac{ns(1+s)(n+\mu)^2}{(1+\mu)(1-s)}e^{-smH}\right]$$

$$= \frac{s(n+\mu)}{(1+\mu)}\frac{1}{1-s}e^{-smH} - \frac{s}{1-s}e^{-mH}$$

During the integration proces, the value of m tends to infinity, so that the limit expression of the equation becomes

$$\lim_{m\to\infty}\left[Am^2(1+\mu)^2e^{mH} + Csm^2s(n+\mu)^2e^{smH}\right] = 0$$

This relation can only be satisfied by setting

$$C = 0 \quad \text{when} \quad s > 1$$
$$A = 0 \quad \text{when} \quad s < 1$$

It is easily shown that in the case of an isotropic body, one must always have C = 0, because of the factor z multiplying C in the stressfunction.

Taking $s<1$, C becomes

$$ns(n+\mu)Csm^2 = \frac{e^{-smH}\left[(1+\mu)e^{-mH} - (n+\mu)e^{-smH}\right]}{2(1+\mu)e^{-mH}\cdot e^{-smH} - (n+\mu)(1-s) - (1+s)(n+\mu)e^{-2smH}}$$

The expression of the deflection at the surface and in the axle of the load$[J_0(mr) = 1 \text{ for } r=0]$ is

$$W = \frac{1+\mu}{E}\cdot pa\int_0^\infty \frac{J_1(ma)}{m}\left[1 - 2ns(n+\mu)Csm^2\right]\frac{s(1-n)}{(1+\mu)(1-s)}dm$$

1.2.3. Fixed bottom by a geometrical and a mechanical condition.

The vertical deflection at a depth z is given by

$$W_1 = \frac{1+\mu}{E}\cdot pa\int_0^\infty \frac{J_0(mr)J_1(ma)}{m}\left[n(1+\mu)(Am^2e^{mz} - Bm^2e^{-mz})\right.$$

$$\left. + \frac{ns(n+\mu)^2}{(1+\mu)}(Csm^2e^{smz} - Dsm^2e^{-smz})\right]dm$$

But with the general solutions of the compatibility equations in multilayer theory, we can do it also in another way by expressing that w = 0 at the desired depth and determinating the corresponding values of the parameters A_n, B_n, C_n and D_n.

Since there are thus several possibilities to express a same boundary condition, it is necessary to compare the results obtained and retain the one that seems the most appropriate.

To do this we shall consider the most simple case, that of the semi-infinite body: the one-layer case.

I.2.1. Basic equations.

Let us consider a semi-infinite anisotropic body submitted to a uniformly distributed vertical pressure at its surface.

The stressfunction is

$$\phi = pa \int_0^{\infty} \frac{J_0(mr).\, J_1(ma)}{m} \left[Ae^{mz} - Be^{-mz} + Ce^{msz} - De^{-msz} \right] dm$$

The surface boundary conditions ($\sigma_z = p$, $\tau_{rz} = 0$, $z = 0$) are deduced from the relations given in § 1. for the stresses

$$n(1+\mu)(Am^2 + Bm^2) + n(n+\mu)(Csm^2 + Dsm^2) = 1$$

$$n(1+\mu)(Am^2 - Bm^2) + ns(n+\mu)(Csm^2 - Dsm^2) = 0$$

Solving this system for B and D, one obtains

$$n(1-s)(1+\mu)\, Bm^2 = -s + n(1+s)(1+\mu)\, Am^2 + 2ns(n+\mu)\, Csm^2$$

$$n(1-s)(n+\mu)\, Dsm^2 = 1 - 2n(1+\mu)\, Am^2 - n(1+s)(n+\mu)\, Csm^2$$

The next step depends on the boundary condition that fixes the deflection at a depth H.

1.2. The fixed bottom condition.

The boudary conditions discussed in previous alinea implicate that the last layer of the multilayer is considered as a semi-infinite body.
One can also consider the case of a multilayer built on an undeformable body, that thus any vertical displacement vanishes at the contact face with the undeformable body: we shall call this a fixed bottom condition.
This condition can be introduced in several manners and thus demands a detailed analysis.
A vertical displacement is obtained by integration of the vertical strain:

$$w = \int \varepsilon_z \cdot dz$$

It would not be correct to resort to an integration between limits, such as

$$w = \int_0^H \varepsilon_z \cdot dz$$

where H could, for example, be the depth at which we want the bottom to be fixed. In doing so, we would ignore the contribution (zero or not) to the vertical deflection (or displacement) due to other parts of the body that we neglect by integrating between specified limits.
The correct way consists in writing (TIMOSHENKO, 1970):

$$w = \int \varepsilon_z \cdot dz + f(r)$$

where $f(r)$ is a fonction of r only, and thus a constant regarding z, so that by differentiating we obtain again

$$\frac{\partial w}{\partial z} = \varepsilon_z$$

By choosing an appropriate expression for $f(r)$, we can obtain the bottom fixed at the desired depth; by doing this, we introduce in fact a geometrical condition fixing the reference level for the vertical deflections at the choosen depth.

Let us consider an average value of $tg\varphi = 1.5$, value utilized in the design of continuously reinforced concrete pavements (Mc CULLOUGH, 1981).
We deduce from figure 1 the limit values for k:

H/a = 1	k = 0.20
H/a = 2	k = 0.17
H/a = 3	k = 0.10
H/a = 4	k = 0.03

If we consider a surface layer built on a sand basecourse ($\varphi = 30°$, $tg\varphi = 0.58$), the limit values become:

H/a = 1	k = 1
H/a = 2	k = 0.60
H/a = 3	k = 0.40
H/a = 4	k = 0.25

With our approach, the case of full continuity ($u_i = u_{i+1}$, often called full friction) becomes a particular case for which the angle of friction between the layers corresponds with a value of k = 1. For other values of k we have partial continuity ($u_i \neq u_{i+1}$, what could be called partial friction or partial adhesion).
We notice that for values of k smaller than one, u_i is smaller than u_{i+1}. This means practically that the lateral movements of the surface layer are retained, for example by shoulders. Such a construction reduces the vertical stresses on the subground which improves the lifetime of the road structure.
For values of k larger than one, u_i is also larger than u_{i+1}. Here the lateral movements of the surface layer are easier than those of the sublayer, as in overlayconstructions for example.
For the limit value $k = \infty$, one obtains also $\tau_{rz} = 0$.
This case is called the frictionless interface.
For $k = \infty$, one obtains indeed

$$\sigma_z = 2pa\int_0^\infty J_0(mr)J_1(ma)\frac{\left[(1+mH)e^{-mH} - (1-mH)e^{-3mH}\right].E_2}{(E_1-E_2)e^{-4mH} + 2\left[2mH.E_2 - E_1 - 2m^2H^2.E_1\right]e^{-2mH} + (E_1+E_2)}\cdot dm$$

which is the value for σ_z in the case of a frictionless interface (BURMISTER, 1943) as will be shown in § 2.1.

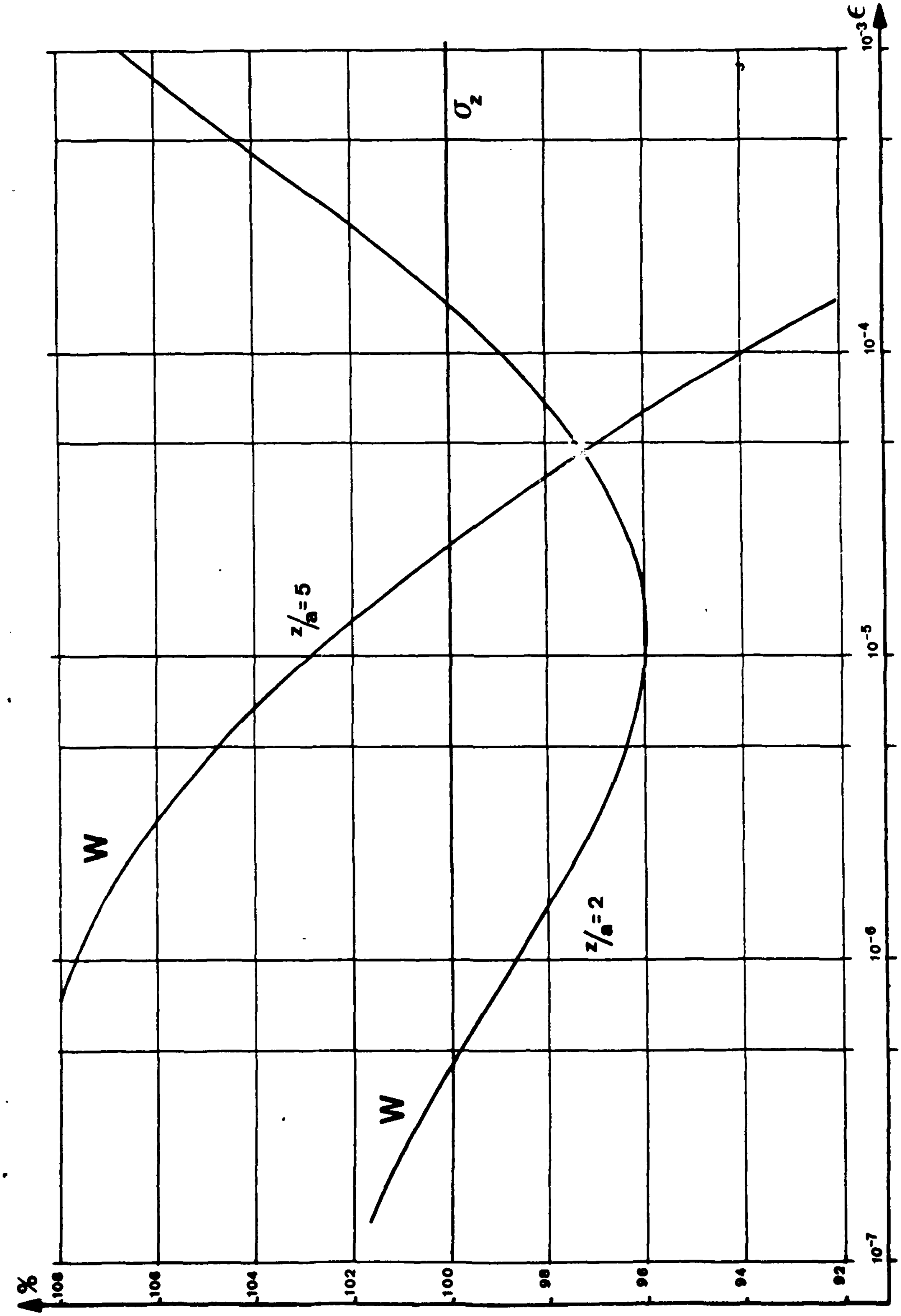


Figure 2

The only way to solve the problem satisfactory is to split the expression of the deflection in another way than the one we had done.
We first write the expression of the deflection with negative exponents only:

$$w = -pa\,\frac{2(1-\mu_1^2)}{E_1}\int_0^\infty \frac{J_0(mr).J_1(ma)}{m}\,.$$

$$\left\{\frac{F \;-(2F-1-2mH)e^{-2mH}-(1-F)\,e^{-4mH}}{F+\left[(2F-1)\,2mH-(1+2m^2H^2)\right]e^{-2mH}+(1-F)e^{-4mH}}\right\}dm$$

We then divide the numerator of the term in between brackets by the denominator:

$$w = -pa\,\frac{2(1-\mu_1^2)}{E_1}\int_0^\infty \frac{J_0(mr).J_1(ma)}{m}\,.$$

$$\left\{1 \;+\; 2\,\frac{\left[(1-F)(1+mH)+m^2H^2\right]e^{-2mH}-(1-F)\,e^{-4mH}}{F+\left[(2F-1)2mH-(1+2m^2H^2)\right]e^{-2mH}+(1-F)\,e^{-4mH}}\right\}dm$$

and split the integral into two parts from which the first is integrable analytically and the second converges in the usual way.

For $r = 0$, one has

$$\int_0^\infty \frac{J_1(ma)}{m}\,dm = 1$$

For $r = a$, one has

$$\int_0^\infty \frac{J_0(ma).J_1(ma)}{m}\,dm = 2/\pi$$

For $r < a$, one has

$$\int_0^\infty \frac{J_0(mr).J_1(ma)}{m}\,dm = F\left(\tfrac{1}{2}, -\tfrac{1}{2}; 1; \frac{r^2}{a^2}\right)$$

where F is the hypergeometric function of GAUSS:

$$F(a, b; c; z) = \sum_0^\infty \frac{(a)_n(b)_n}{n!\,(c)_n}\,z^n$$

$$(a)_n = a(a+1)(a+2)\cdots(a+n-1) \qquad\qquad (a)_0 = 1$$

For $r > a$, one has

$$\int_0^\infty \frac{J_0(mr)\cdot J_1(ma)}{m}\,dm = \frac{a}{2r}\,F\left(\frac{1}{2},\frac{1}{2};2;\frac{a^2}{r^2}\right)$$

The obtained result will now be correct, while convergency is reached as fast as for the other equations for stresses.
But again, if we are to be able to compute as indicated, we must have the equations in closeform at our disposal, although in such a form that the integral can be split.

2.4. Convergency in the first layer.

As for the deflection at the surface, the numerical computation of the stresses in the first layer, in fact nearby the surface, also converges very slowly.

To illustrate this let us look at the relation for the vertical stress in the first layer (the equation is given in § 1.):

$$\sigma_z = pa \int_0^\infty J_0(mr)\, J_1(ma) \left[A_1 m^2 e^{mz} + B_1 m^2 e^{-mz} - C_1 m (1-2\mu_1 - mz) e^{mz} + D_1 m (1-2\mu_1 + mz) e^{-mz} \right] dm$$

We replace A_1, B_1, C_1, D_1 by their values obtained in § 2.1.

$$\sigma_z = pa \int_0^\infty J_0(mr)\, J_1(ma) \cdot \left\{ F\left[1+m(H+z)\right] e^{-mH} \cdot e^{-mz} + \left[\quad \right] e^{-mH} \cdot e^{mz} + \left[\quad \right] e^{-3mH} \cdot e^{mz} + \left[\quad \right] e^{-3mH} \cdot e^{-mz} \right\} \frac{dm}{\nabla}$$

The values of z are negative in the first layer (z = 0 at the interface).

The term

$$F\left[1+m(H+z)\right] e^{-mH} \cdot e^{-mz} \cdot \frac{1}{\nabla}$$

converges very slowly for values of -z nearly equal to H, the other terms converge normally.

To solve the problem created by the first term we divide again numerator by denominator:

$$F\left[1+m(H+z)\right] e^{-mH} \cdot e^{-mz} \cdot \frac{1}{\nabla} = \left[1+m(H+z)\right] e^{-mH} \cdot e^{-mz} - \frac{\left[1+m(H+z)\right] e^{-mH} \cdot e^{-mz} \left\{ \left[(2F-1)2mH - (1+2m^2H^2)\right] e^{-2mH} + (1-F) e^{-4mH} \right\}}{F + \left[(2F-1)2mH - (1+2m^2H^2)\right] e^{-2mH} + (1-F) e^{-4mH}}$$

The second term of the second member converges normally so that we can again split the integral in several parts from which the one that converges slowly is

$$pa \int_0^\infty J_0(mr) \cdot J_1(ma) \left[1+m(H+z)\right] e^{-mH} \cdot e^{-mz} dm.$$

This integral is known as a LIPSCHITZ-HANKEL integral, but only some particular cases are integrable analytically. To solve the problem for all cases we have to make a detourthrough the analysis of stresses and displacements in a semi-infinite body submitted to an isolated local force P.

2.4.1. Stresses and displacements under an isolated load.

The Hankel transform in the case of a uniformly distributed load is

$$\psi(m) = pa \int_0^\infty \frac{J_1(am)}{m} \phi \, dm$$

We consider the resulting load $P = \pi pa^2$ acting on a surface whose area reduces to zero.

$$\psi(m) = \lim_{a \to 0} \frac{P}{\pi} \int_0^\infty \frac{J_1(am)}{am} \phi \, dm$$

$$= \frac{P}{2\pi} \int_0^\infty \phi \, dm$$

The relations for the stresses and the displacements are then given, in the case of an isotropic body, by

$$\sigma_r = \frac{P}{2\pi} \Big[\int_0^\infty m J_0(mr) e^{-mz} dm - z \int_0^\infty m^2 J_0(mr) e^{-mz} dm$$
$$- (1-2\mu) \int_0^\infty \frac{J_1(mr)}{r} e^{-mz} dm + z \int_0^\infty m \frac{J_1(mr)}{r} e^{-mz} dm \Big]$$

$$= \frac{P}{2\pi} \Big\{ \frac{z}{(z^2+r^2)^{3/2}} - \Big[\frac{2z}{(z^2+r^2)^{3/2}} - \frac{3zr^2}{(z^2+r^2)^{5/2}} \Big]$$
$$- (1-2\mu) \frac{(r^2+z^2)^{1/2} - z}{r^2 (r^2+z^2)^{1/2}} + \frac{z}{(z^2+r^2)^{3/2}} \Big\}$$

$$\sigma_z = \frac{P}{2\pi} \Big[\int_0^\infty m J_0(mr) e^{-mz} dm + z \int_0^\infty m^2 J_0(mr) e^{-mz} dm \Big]$$

$$= \frac{P}{2\pi} \Big\{ \frac{z}{(z^2+r^2)^{3/2}} + \Big[\frac{2z}{(z^2+r^2)^{3/2}} - \frac{3zr^2}{(z^2+r^2)^{5/2}} \Big] \Big\}$$

$$\tau_{rz} = \frac{Pz}{2\pi}\int_0^\infty m^2 J_1(mr)\, e^{-mz}\, dm$$

$$= \frac{P}{2\pi}\,\frac{3z^2 r}{(r^2+z^2)^{3/2}}$$

$$W = -\frac{P}{2\pi}\,\frac{2(1-\mu^2)}{E}\int_0^\infty J_0(mr)\, e^{-mz}\, dm - \frac{Pz}{2\pi}\,\frac{(1+\mu)}{E}\int_0^\infty m\, J_0(mr)\, e^{-mz}\, dm$$

$$= -\frac{P}{2\pi}\,\frac{2(1-\mu^2)}{E}\,\frac{1}{(z^2+r^2)^{1/2}} - \frac{P}{2\pi}\,\frac{(1+\mu)}{E}\,\frac{z}{(z^2+r^2)^{3/2}}$$

$$u = \frac{P}{2\pi}\,\frac{(1+\mu)(1-2\mu)}{E}\int_0^\infty J_1(mr)\, e^{-mz}\, dm - \frac{Pz}{2\pi}\,\frac{(1+\mu)}{E}\int_0^\infty m\, J_1(mr)\, e^{-mz}\, dm$$

$$= \frac{P}{2\pi}\,\frac{(1+\mu)(1-2\mu)}{E}\cdot\frac{(r^2+z^2)^{1/2}-z}{r(r^2+z^2)^{1/2}} - \frac{P}{2\pi}\,\frac{(1+\mu)}{E}\cdot\frac{zr}{(z^2+r^2)^{3/2}}$$

2.4.2. Stresses and displacements under a uniformly distributed load.

The stresses and displacements under a uniformly distributed load can be obtained by integrating the relations under an isolated load over the concerned area

$$\sigma_D = \int_0^{2\pi}\int_0^a \sigma_I\, \rho\, d\rho\, d\theta$$

where σ_I is given by one of the relations of § 2.4.1. wherein the distance r must be replaced by $(r^2 + \rho^2 - 2r\rho\cos\theta)^{1/2}$

2.4.3. Relations for the stress σ_r

The relation for σ_r under a distributed load is given by

$$\sigma_r = pa\int_0^\infty J_0(mr)\cdot J_1(ma)\, e^{-mz}\, dm - pa\int_0^\infty J_0(mr)\cdot J_1(ma)\, mz\, e^{-mz}\, dm$$
$$- pa\,(1-2\mu)\int_0^\infty \frac{J_1(mr)}{mr}\, J_1(ma)\, e^{-mz}\, dm + pa\int_0^\infty \frac{J_1(mr)}{r}\, J_1(ma)\, z\, e^{-mz}\, dm$$

so that comparing with the expression for the stress under an isolated load we can conclude that

$$I_1 : \; pa\int_0^\infty J_0(mr)\,J_1(ma)\,e^{-mz}\,dm = \frac{p}{2\pi}\int_0^{2\pi}\!\!\int_0^a \frac{z\rho}{(z^2+r^2+\rho^2-2r\rho\cos\theta)^{3/2}}\,d\rho\, d\theta$$

$$I_2 : \; pa\int_0^\infty J_0(mr)\,J_1(ma)\,mz\,e^{-mz}\,dm =$$

$$\frac{p}{2\pi}\int_0^{2\pi}\!\!\int_0^a \left[\frac{2z\rho}{(z^2+r^2+\rho^2-2r\rho\cos\theta)^{3/2}} - \frac{3z(r^2+\rho^2-2r\rho\cos\theta)\cdot\rho}{(z^2+r^2+\rho^2-2r\rho\cos\theta)^{5/2}}\right] d\rho\, d\theta$$

$$I_3 : \; pa\int_0^\infty \frac{J_1(mr)}{mr}\,J_1(ma)\,e^{-mz}\,dm = \frac{p}{2\pi}\int_0^{2\pi}\!\!\int_0^a \frac{(z^2+r^2+\rho^2-2r\rho\cos\theta)^{1/2}-z}{(r^2+\rho^2-2r\rho\cos\theta)(z^2+r^2+\rho^2-2r\rho\cos\theta)^{1/2}}\cdot\rho\, d\rho\, d\theta$$

$$I_4 : \; pa\int_0^\infty \frac{J_1(mr)}{r}\cdot J_1(ma)\,z\,e^{-mz}\,dm = \frac{p}{2\pi}\int_0^{2\pi}\!\!\int_0^a \frac{z\rho\, d\rho\, d\theta}{(z^2+r^2+\rho^2-2r\rho\cos\theta)^{3/2}}$$

2.4.4. Relations for the stress σ_z

These relations can be deduced from those established for the stress σ_r

2.4.5. Relation for the stress τ_{rz}

$$\tau_{rz} = pa\int_0^\infty J_1(mr)\cdot J_1(ma)\,mz\,e^{-mz}\,dm$$

$$I_5 : \; pa\int_0^\infty J_1(mr)\cdot J_1(ma)\,mz\,e^{-mz}\,dm = \frac{p}{2\pi}\int_0^{2\pi}\!\!\int_0^a \frac{3z^2(r^2+\rho^2-2r\rho\cos\theta)^{1/2}}{(z^2+r^2+\rho^2-2r\rho\cos\theta)^{5/2}}\,\rho\, d\rho\, d\theta$$

2.4.6. Relations for the vertical displacement w.

$$w = -pa\,\frac{2(1-\mu^2)}{E}\int_0^\infty \frac{J_0(mr)\cdot J_1(ma)}{m}\cdot e^{-mz}\,dm$$

$$-pa\,\frac{(1+\mu)}{E}\int_0^\infty J_0(mr)\cdot J_1(ma)\,z\,e^{-mz}\,dm$$

$$I_6 : \; pa\int_0^\infty \frac{J_0(mr)\cdot J_1(ma)}{m}\,e^{-mz}\,dm = \frac{p}{2\pi}\int_0^{2\pi}\!\!\int_0^a \frac{\rho\, d\rho\, d\theta}{(z^2+r^2+\rho^2-2r\rho\cos\theta)^{1/2}}$$

$$I_7 : \; pa\int_0^\infty J_0(mr)\cdot J_1(ma)\,z\,e^{-mz}\,dm = \frac{p}{2\pi}\int_0^{2\pi}\!\!\int_0^a \frac{z^2\rho\cdot d\rho\cdot d\theta}{(z^2+r^2+\rho^2-2r\rho\cos\theta)^{3/2}}$$

2.4.7. Relations for the horizontal displacement u.

$$u = pa\,\frac{(1+\mu)(1-2\mu)}{E}\int_0^\infty \frac{J_1(mr)\,J_1(ma)}{m}\,e^{-mz}\,dm$$

$$-\,pa\,\frac{(1+\mu)}{E}\int_0^\infty J_1(mr)\cdot J_1(ma)\,z\,e^{-mz}\,dm$$

$$I_8 : pa\int_0^\infty \frac{J_1(mr)\,J_1(ma)}{m}\,e^{-mz}\,dm = \frac{p}{2\pi}\int_0^{2\pi}\int_0^{a}\frac{\left[(z^2+r^2+\rho^2-2r\rho\cos\theta)^{1/2}-z\right]\rho\,d\rho\,d\theta}{(r^2+\rho^2-2r\rho\cos\theta)^{1/2}\,(z^2+r^2+\rho^2-2r\rho\cos\theta)^{1/2}}$$

$$I_9 : pa\int_0^\infty J_1(mr)\cdot J_1(ma)\,z\,e^{-mz}\,dm = \frac{p}{2\pi}\int_0^{2\pi}\int_0^{a}\frac{z\,(r^2+\rho^2-2r\rho\cos\theta)^{1/2}\rho\,d\rho\,d\theta}{(z^2+r^2+\rho^2-2r\rho\cos\theta)^{3/2}}$$

2.4.8. Resolution of the double integrals.

The integrals of alinea 2.4.7. are most easily solved in transforming the variables θ and ρ by setting

$$x=\rho\cos\theta \qquad y=\rho\sin\theta \qquad \rho\,d\rho\,d\theta = dx\,dy$$

One obtains

$$I_1 = \frac{p}{2\pi}\int_{-a}^{a}\frac{2z(a^2-x^2)^{1/2}}{(z^2+x^2+r^2-2xr)\,(z^2+r^2+a^2-2xr)^{1/2}}\,dx$$

that can easily be computed numerically.

$$I_2 = \frac{p}{2\pi}\int_{-a}^{a}\frac{2z\,(r^2+x^2-2rx)(a^2-x^2)^{1/2}}{(z^2+x^2+r^2-2xr)\,(z^2+r^2+a^2-2xr)^{1/2}}\left[\frac{1}{(r^2+a^2-2xr+z^2)}+\frac{2}{(z^2+x^2+r^2-2rx)}\right]dx$$

$$+\,\frac{p}{2\pi}\int_{-a}^{a}\frac{2z(a^2-x^2)^{3/2}}{(z^2+x^2+r^2-2xr)(z^2+r^2+a^2-2xr)^{3/2}}\,dx$$

$$-\,\frac{p}{2\pi}\int_{-a}^{a}\frac{4z(a^2-x^2)^{1/2}}{(z^2+x^2+r^2-2rx)(z^2+r^2+a^2-2rx)^{1/2}}\,dx$$

I_3 and I_4 are particular Lipschitz-Hankel integrals that we shall solve in next alinea.

I_5 can be deduced from I_2.

$$I_6 = \frac{p}{2\pi}\int_{-a}^{a} \ln \frac{(r^2-2xr+a^2+z^2)^{1/2} + (a^2-x^2)^{1/2}}{(r^2-2xr+a^2+z^2)^{1/2} - (a^2-x^2)^{1/2}}\, dx$$

I_7 can be deduced from I_1.

I_8 and I_9 are particular Lipschitz-Hankel integrals.

2.4.9. Resolution of the Lipschitz-Hankel integrals.

The solution of the Lipschitz-Hankel integrals is given by WATSON (1960).

$$\int_0^{\infty} e^{-at} J_\nu(bt) . J_\mu(ct)\, t^{\mu-1} dt =$$

$$\frac{(bc)^\nu\, \Gamma(\mu+2\nu)}{\pi\, a^{\mu+2\nu}\, \Gamma(2\nu+1)} \int_0^{\pi} F\left(\frac{\mu+2\nu}{2}, \frac{\mu+2\nu+1}{2}; \nu+1; -\frac{\omega^2}{a^2}\right) \sin^{2\nu}\phi\, d\phi$$

where

$$\omega^2 = b^2 + c^2 - 2bc\cos\phi$$

$$I_3 = p\frac{a}{r}\int_0^{\infty} \frac{J_1(mr) . J_1(ma)}{m} e^{-mz} dm$$

$$= p\,\frac{a^2}{z^2}\cdot\frac{1}{2\pi}\int_0^{\pi} F\left(1, 3/2; 2; -\frac{\omega^2}{z^2}\right) \sin^2\phi\, d\phi$$

with (WAYLAND,1970)

$$F\left(1, 3/2; 2; -\frac{\omega^2}{z^2}\right) = \frac{z^2}{\omega^2}\left[1 - \left(1 + \frac{\omega^2}{z^2}\right)^{-1/2}\right]$$

and $\quad \omega^2 = a^2 + r^2 - 2ar\cos\phi$

The resulting integral can easily be solved numerically.

$$I_4 = \frac{p a z}{r} \int_0^\infty J_1(mr) \cdot J_1(ma)\, dm$$

$$= p \frac{a^2}{z^2} \cdot \frac{1}{\pi} \int_0^\pi F\left(\frac{3}{2}, 2; 2; -\frac{\omega^2}{z^2}\right) \sin^2\phi\, d\phi$$

$$F\left(\frac{3}{2}, 2; 2; -\frac{\omega^2}{z^2}\right) = \left(1 + \frac{\omega^2}{z^2}\right)^{-3/2}$$

I_8 can be deduced from I_3

I_9 can be deduced from I_4:

2.4.10. Expressions for computations in the axle of the load.

When stresses and displacements are computed in the axle of the load, one has

$$I_1 : \ p a \int_0^\infty J_1(ma)\, e^{-mz}\, dm = \frac{\sqrt{a^2+z^2} - z}{a \cdot \sqrt{a^2+z^2}}$$

$$I_2 : \ p a \int_0^\infty J_1(ma)\, mz\, e^{-mz}\, dm = \frac{a z}{(a^2+z^2)^{3/2}}$$

$$I_3 : \ p a \int_0^\infty \frac{J_1(mr) \cdot J_1(ma)}{mr}\, e^{-mz}\, dm = \frac{1}{2} \cdot p a \int_0^\infty J_1(ma)\, e^{-mz}\, dm = \frac{I_1}{2}$$

$$I_4 : \ p a \int_0^\infty \frac{J_1(mr)}{r} \cdot J_1(ma)\, z\, e^{-mz}\, dm = \frac{1}{2} p a \int_0^\infty J_1(ma)\, mz\, e^{-mz}\, dm = \frac{I_2}{2}$$

$$I_5 : \ p a \int_0^\infty J_0(mr)\, J_1(ma)\, mz\, e^{-mz}\, dm = 0$$

$$I_6 = p a \int_0^\infty \frac{J_1(ma)}{m}\, e^{-mz}\, dm = \frac{\sqrt{a^2+z^2} - z}{a}$$

$$I_7 : \ p a \int_0^\infty J_1(ma)\, z\, e^{-mz}\, dm = z \cdot I_1$$

$$I_8 : \ p a \int_0^\infty \frac{J_1(mr)\, J_1(ma)}{m}\, e^{-mz}\, dm = 0$$

$$I_9 : \ p a \int_0^\infty J_1(mr)\, J_1(ma)\, z\, e^{-mz}\, dm = 0$$

To be applicable, all the developments of paragraph 2.4. again require all the equations to be available in closeform.

3. The complete algebraical solution.

We shall develop here the complete algebraical analysis leading to the different computerprograms in which, because of the particular method that we have adopted, all the problems detailled above are solved.
But to make the understanding of the method easier we first develop the analysis, with the necessary comments, for a three-layer system.
This will enable us in the same time to verify the results obtained with our original method with those of existing programs.

3.1. Algebraical analysis of a three-layer system (isotropic, full friction).

3.1.1. Boundary conditions of the system.

We consider $\mu_1 = \mu_2 = \mu_3 = 0.5$

H_1 the thickness of the first layer

H_2 the thickness of the second layer

We write A_1 for $A_1 m^2$

B_1 for $B_1 m^2$

C_1 for $C_1 m$

D_1 for $D_1 m$

and $$K = \frac{E_1(1+\mu_2)}{E_2(1+\mu_1)} = \frac{E_1}{E_2}$$

$$L = \frac{E_2(1+\mu_3)}{E_3(1+\mu_2)} = \frac{E_2}{E_3}$$

Boundary conditions at the surface ($z = 0$):

$\sigma_z = p$: $A_1 + B_1 = 1$

$\tau_{rz} = 0$: $A_1 - B_1 + C_1 + D_1 = 0$

Boundary conditions at the first interface ($z = H_1$, $x = mH_1$):

σ_z : $A_1 e^x + B_1 e^{-x} + x C_1 e^x + x D_1 e^{-x} = A_2 e^x + B_2 e^{-x} + x C_2 e^x + x D_2 e^{-x}$

τ_{rz} : $A_1 e^x - B_1 e^{-x} + C_1(1+x)e^x + D_1(1-x)e^{-x} = A_2 e^x - B_2 e^{-x} + C_2(1+x)e^x + D_2(1-x)e^{-x}$

w : $A_1 e^x - B_1 e^{-x} + x C_1 e^x - x D_1 e^{-x} = K[A_2 e^x - B_2 e^{-x} + x C_2 e^x - x D_2 e^{-x}]$

u : $A_1 e^x + B_1 e^{-x} + C_1(1+x)e^x - D_1(1-x)e^{-x} = k[A_2 e^x + B_2 e^{-x} + C_2(1+x)e^x - D_2(1-x)e^{-x}]$

Boundary conditions at the second interface ($z = H_1 + H_2$, $y = m.(H_1 + H_2)$):

$$\sigma_z: \; A_2e^{y} + B_2e^{-y} + yC_2e^{y} + yD_2e^{-y} = B_3e^{-y} + yD_3e^{-y}$$
$$\tau_{rz}: \; A_2e^{y} - B_2e^{-y} + C_2(1+y)e^{y} + D_2(1-y)e^{-y} = -B_3e^{-y} - D_3(1-y)e^{-y}$$
$$w: \; A_2e^{y} - B_2e^{-y} + yC_2e^{y} - yD_2e^{-y} = L\left[-B_3e^{-y} - yD_3e^{-y}\right]$$
$$u: \; A_2e^{y} + B_2e^{-y} + C_2(1+y)e^{y} - D_2(1-y)e^{-y} = L\left[B\, e^{-y} - D_3(1-y)e^{-y}\right]$$

The boundary conditions can be written in matrixform

At the surface

$$I\,(A_1\; B_1\; C_1\; D_1)^T = (1 \quad 0)^T$$

where

$$I = \begin{pmatrix} 1 & 1 & 0 & 0 \\ 1 & -1 & 1 & 1 \end{pmatrix}$$

At the first interface

$$M_1\,(A_1\, B_1\, C_1\, D_1)^T = M_2\,(A_2 B_2 C_2 D_2)^T$$

At the second interface

$$M_3\,(A_2 B_2 C_2 D_2)^T = M_4\,(B_3\; D_3)^T$$

3.1.2. Solution of the system of 10 equations.

We start from the conditions at the second interface and write

$$(A_2 B_2 C_2 D_2)^T = M_3^{-1} \cdot M_4\,(B_3\; D_3)^T \qquad (1)$$

Matrix M_3 is very easy to invert:

$$M_3^{-1} = -\frac{1}{4}\begin{pmatrix} -2(1+y)e^{-y} & 2ye^{-y} & -2(1+y)e^{-y} & 2ye^{-y} \\ -2(1-y)e^{+y} & 2ye^{y} & 2(1-y)e^{y} & -2ye^{y} \\ 2e^{-y} & -2e^{-y} & 2e^{-y} & -2e^{-y} \\ -2e^{y} & -2e^{y} & 2e^{y} & 2e^{y} \end{pmatrix}$$

that we write as follows

$$M_3^{-1} = -\frac{1}{2}\left[e^{-y} \cdot M_{31} + e^{y} M_{32}\right]$$

with

$$M_{31} = \begin{pmatrix} -(1+y) & y & -(1+y) & y \\ 0 & 0 & 0 & 0 \\ 1 & -1 & 1 & -1 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

$$M_{32} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ -(1-y) & y & (1-y) & -y \\ 0 & 0 & 0 & 0 \\ -1 & -1 & 1 & 1 \end{pmatrix}$$

and

$$M_4 = e^{-y} . M_{41}$$

with

$$M_{41} = \begin{pmatrix} 1 & y \\ -1 & (1-y) \\ -L & -Ly \\ L & -L(1-y) \end{pmatrix}$$

so that

$$(A_2 B_2 C_2 D_2)^T = -\frac{1}{2}\left[e^{-2y} . M_{31} . M_{41} + M_{32} . M_{41}\right](B_3 \; D_3)^T \qquad (2)$$

We notice that the positive exponent e^y has disappeared

We now develop those matrixproducts necessary to avoid convergency and overflowproblems. For the others we only take notice of the terms equal to zero.

So that

$$M_{31} . M_{41} = \begin{pmatrix} + & + \\ 0 & 0 \\ + & + \\ 0 & 0 \end{pmatrix} = M_{3141}$$

where the sign + denotes some constant or some linear function of m.

$$M_{32}.M_{41} = -(1+L)\begin{pmatrix} 0 & 0 \\ 1 & 0 \\ 0 & 0 \\ 0 & 1 \end{pmatrix} = -(1+L).U_L$$

and

$$(A_2 B_2 C_2 D_2)^T = -\frac{1}{2}\left[e^{-2y}.M_{3141} - (1+L).U_L\right](B_3 D_3)^T \qquad (3)$$

The term $(1 + L).U_L$ will be part of the constant in the final denominator, constant whose value we have to know to ensure convergency at the surface.

We now consider the conditions at the first interface:

$$(A_1 B_1 C_1 D_1)^T = M_1^{-1}.M_2\,(A_2 B_2 C_2 D_2)^T \qquad (4)$$

Matrix M_1^{-1} is identic to matrix M_3^{-1} in which y is replaced by x:

$$M_1^{-1} = -\frac{1}{2}\left[e^{-x}.M_{11} + e^{x}M_{12}\right]$$

We write matrix M_2 as follows

$$M_2 = e^{x}\begin{pmatrix} 1 & 0 & x & 0 \\ 1 & 0 & (1+x) & 0 \\ k & 0 & kx & 0 \\ k & 0 & k(1+x) & 0 \end{pmatrix} + e^{-x}\begin{pmatrix} 0 & 1 & 0 & x \\ 0 & -1 & 0 & (1-x) \\ 0 & -k & 0 & -1+x \\ 0 & k & 0 & -k(1-x) \end{pmatrix}$$

$$= e^{x}M_{21} + e^{-x}M_{22}$$

and

$$(A_1 B_1 C_1 D_1)^T = -\frac{1}{2}\left[(M_{11}.M_{21} + M_{12}.M_{22}) + e^{-2x}M_{11}.M_{22} + e^{2x}M_{12}.M_{21}\right] .(A_2 B_2 C_2 D_2)^T \qquad (5)$$

We again develop the products as before:

$$M_{31} \cdot M_{42} = \begin{pmatrix} 0 & + & 0 & + \\ 0 & 0 & 0 & 0 \\ 0 & + & 0 & + \\ 0 & 0 & 0 & 0 \end{pmatrix} = M_{3142}$$

$$M_{32} \cdot M_{41} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ + & 0 & + & 0 \\ 0 & 0 & 0 & 0 \\ + & 0 & + & 0 \end{pmatrix} = M_{3241}$$

$$(A_2 B_2 C_2 D_2)^T = \frac{1}{16(1-\mu_2)(1-\mu_3)} \left[e^{-2y} M_{3142} + e^{2y} M_{3241} + (M_{31} \cdot M_{41} + M_{32} \cdot M_{42}) \right]$$

$$\cdot \left[e^{-2t} M_{5161} + e^{-2z} M_{5162} + e^{-2(t-z)} M_{5261} + M_{52} \cdot M_{62} \right] (B_4 \; D_4)^T$$

The products which will not converge normally are

$$e^{2y} \cdot e^{-2(t-z)} \cdot M_{3241} \cdot M_{5261}$$

because it can happen that $y > t - z$

and

$$e^{2y} \cdot M_{3241} \cdot M_{52} \cdot M_{62}$$

One easily verifies that

$$M_{3241} \cdot M_{5261} = 0$$

$$M_{3241} \cdot M_{52} \cdot M_{62} = 0$$

so that the concerned products disappear from the relation.

One verifies also that

$$M_{3142} \cdot M_{5161} = 0$$

$$M_{3142} \cdot M_{5162} = 0$$

$$M_4 = e^{y}\begin{pmatrix} 1 & 0 & -(1-2\mu_3-y) & 0 \\ 1 & 0 & (2\mu_3+y) & 0 \\ k & 0 & -k(2-4\mu_3-y) & 0 \\ k & 0 & k(1+y) & 0 \end{pmatrix}$$

$$+\, e^{-y}\begin{pmatrix} 0 & 1 & 0 & (1-2\mu_3+y) \\ 0 & -1 & 0 & (2\mu_3-y) \\ 0 & -k & 0 & -k(2-4\mu_3+y) \\ 0 & k & 0 & -k(1-y) \end{pmatrix}$$

$$(A_2 B_2 C_2 D_2)^T = -\frac{1}{4(1-\mu_2)}\left[e^{-y} M_{31} + e^{y} M_{32}\right]\left[e^{y} M_{41} + e^{-y} M_{42}\right](A_3 B_3 C_3 D_3)^T$$

$$= -\frac{1}{4(1-\mu_2)}\cdot\left[M_{31}\cdot M_{41} + e^{-2y} M_{31}\cdot M_{42} + e^{2y} M_{32}\cdot M_{41} + M_{32}\cdot M_{42}\right] \cdot (A_3 B_3 C_3 D_3)^T$$

$$M_{31}\cdot M_{41} + M_{32}\cdot M_{42} =$$

$$\begin{pmatrix} K_1 & 0 & -K_3 & 0 \\ 0 & K_1 & 0 & K_3 \\ 0 & 0 & K_2 & 0 \\ 0 & 0 & 0 & K_2 \end{pmatrix} + y(K_1 - K_2)\begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

The first matrix is a constant, the second is a linear function of y.

$$M_{52}.M_{62} = \begin{pmatrix} 0 & 0 \\ L_1 & L_3 \\ 0 & 0 \\ 0 & L_2 \end{pmatrix} + z(L_1-L_2)\begin{pmatrix} 0 & 0 \\ 0 & 1 \\ 0 & 0 \\ 0 & 0 \end{pmatrix}$$

where

$$L_1 = -2(1-2\mu_3)-(1+L)$$
$$L_2 = -2L(1-2\mu_4)-(1+L)$$
$$L_3 = \frac{1}{2}\left[L_1(1-4\mu_4)-L_2(1-4\mu_3)\right]$$

$$(A_3B_3C_3D_3)^T = -\frac{1}{4(1-\mu_3)}\left[e^{-2t}M_{5161}+e^{-2z}M_{5162}+e^{-2(t-z)}M_{5261}+M_{52}.M_{62}\right] \cdot (B_4D_4)^T \qquad \text{(Sol 1)}$$

The three first terms of the expression between brackets converge normally; the last product $M_{52}.M_{62}$ contains a constant and a term, linear function of z.

We now write the conditions at the second interface in matrixform:

$$(A_2B_2C_2D_2)^T = M_3^{-1}.M_4.(A_3B_3C_3D_3)^T$$

$$M_3^{-1} = -\frac{1}{4(1-\mu_2)}e^{-y}\begin{pmatrix} -(1+y) & -(2-4\mu_2-y) & -(2\mu_2+y) & -(1-2\mu_2-y) \\ 0 & 0 & 0 & 0 \\ 1 & -1 & 1 & -1 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

$$-\frac{1}{4(1-\mu_2)}e^{y}\begin{pmatrix} 0 & 0 & 0 & 0 \\ -(1-y) & (2-4\mu_2+y) & (2\mu_2-y) & -(1-2\mu_2+y) \\ 0 & 0 & 0 & 0 \\ & & & 1 \end{pmatrix}$$

$$M_6 = e^{-u}\begin{pmatrix} 1 & (2-4\mu_4+t) \\ 1 & (2-4\mu_4+t) \\ L & L(2-4\mu_4+t) \\ L & L(2-4\mu_4+t) \end{pmatrix} + e^{-2}\begin{pmatrix} 1 & (1-2\mu_4+2) \\ -1 & (2\mu_4-2) \\ L & -L(2-4\mu_4+2) \\ -L & -L(1-2) \end{pmatrix}$$

$$\left(A_3 B_3 C_3 D_3\right)^T = -\frac{1}{4(1-\mu_3)}\left[e^{-2}\cdot M_{51} + e^{2} M_{52}\right]\cdot\left[e^{-u}\cdot M_{61} + e^{-2} M_{62}\right]$$

$$M_{51}\cdot M_{61} = \begin{pmatrix} + & + \\ 0 & 0 \\ + & + \\ 0 & 0 \end{pmatrix} = M_{5161}$$

$$M_{51}\cdot M_{62} = \begin{pmatrix} + & + \\ 0 & 0 \\ + & + \\ 0 & 0 \end{pmatrix} = M_{5162}$$

$$M_{52}\cdot M_{61} = \begin{pmatrix} 0 & 0 \\ + & + \\ 0 & 0 \\ + & + \end{pmatrix} = M_{5261}$$

$$w:\quad A_3e^{z} - B_3e^{-z} - C_3(2-4\mu_3-z)e^{z} - D_3(2-4\mu_3+z)e^{-z} =$$
$$L_w\left[A_4e^{z} - B_4e^{-z} - C_4(2-4\mu_4-z)e^{z} - D_4(2-4\mu_4+z)e^{-z}\right]$$
$$u:\quad A_3e^{z} + B_3e^{-z} + C_3(1+z)e^{z} - D_3(1-z)e^{-z} =$$
$$L_u\left[A_4e^{z} + B_4e^{-z} + C_4(1+z)e^{z} - D_4(1-z)e^{-z}\right]$$

Boundary conditions at the bottom ($z = H_1 + H_2 + H_3 + H_4$):

$$w:\quad A_4e^{t} - B_4e^{-t} - C_4(2-4\mu_4-t)e^{t} - D_4(2-4\mu_4+t)e^{-t} = 0$$

Introprie: $\quad C_4 = 0.$

3.2.2. Solution of the system of 16 equations:

In the equations of the conditions at the third interface, A_4 is replaced by its value taken from the fixed bottom ndition:

$$A_4e^{t} = B_4e^{-t} + D_4(2-4\mu_4+t)e^{-t}$$

We write the conditions at the third interface in matrix form

$$(A_3\, B_3\, C_3\, D_3)^T = M_5^{-1}\cdot M_6\,(B_4\;D_4)^T$$

For simplicity here, we take $F_w = F_u = F$, $k_w = k_u = k$, $L_w = L_u = L$

so that $\quad k_1 = k_2 = k_3 = 1$

$$M_5^{-1} = -\frac{1}{4(1-\mu_3)}\cdot e^{-z}\begin{pmatrix} -(1+z) & -(2-4\mu_3-z) & -(2\mu_3+z) & -(1-2\mu_3-z) \\ 0 & 0 & 0 & 0 \\ 1 & -1 & 1 & -1 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

$$-\frac{1}{4(1-\mu_3)}\, e^{z}\begin{pmatrix} 0 & 0 & 0 & 0 \\ -(1-z) & (2-4\mu_3+z) & (2\mu_3-z) & -(1-2\mu_3+z) \\ 0 & 0 & 0 & 0 \\ -1 & -1 & 1 & 1 \end{pmatrix}$$

Boundary conditions at the first interface ($z = H_1$):

$$\sigma_z:\quad A_1e^{x} + B_1e^{-x} - C_1(1-2\mu_1-x)e^{x} + D_1(1-2\mu_1+x)e^{-x} =$$
$$A_2e^{x} + B_2e^{-x} - C_2(1-2\mu_2-x)e^{x} + D_2(1-2\mu_2+x)e^{-x}$$

$$\tau_{rz}:\quad A_1e^{x} - B_1e^{-x} + C_1(2\mu_1+x)e^{x} + D_1(2\mu_1-x)e^{-x} =$$
$$A_2e^{x} - B_2e^{-x} + C_2(2\mu_2+x)e^{x} + D_2(2\mu_2-x)e^{-x}$$

$$w:\quad A_1e^{x} - B_1e^{-x} - C_1(2-4\mu_1+x)e^{x} - D_1(2-4\mu_1+x)e^{-x} =$$
$$F_w\left[A_2e^{x} - B_2e^{-x} - C_2(2-4\mu_2-x)e^{x} - D_2(2-4\mu_2+x)e^{-x}\right]$$

$$u:\quad A_1e^{x} + B_1e^{-x} + C_1(1+x)e^{x} - D_1(1-x)e^{-x} =$$
$$F_u\left[A_2e^{x} + B_2e^{-x} + C_2(1+x)e^{x} - D_2(1-x)e^{-x}\right]$$

Boundary conditions at the second interface ($z = H_1 + H_2$):

$$\sigma_z:\quad A_2e^{y} + B_2e^{-y} - C_2(1-2\mu_2-y)e^{y} + D_2(1-2\mu_2+y)e^{-y} =$$
$$A_3e^{y} + B_3e^{-y} - C_3(1-2\mu_3-y)e^{y} + D_3(1-2\mu_3+y)e^{-y}$$

$$\tau_{rz}:\quad A_2e^{y} - B_2e^{-y} + C_2(2\mu_2+y)e^{y} + D_2(2\mu_2-y)e^{-y} =$$
$$A_3e^{y} - B_3e^{-y} + C_3(2\mu_3+y)e^{y} + D_3(2\mu_3-y)e^{-y}$$

$$w:\quad A_2e^{y} - B_2e^{-y} - C_2(2-4\mu_2-y)e^{y} - D_2(2-4\mu_2+y)e^{-y} =$$
$$K_w\left[A_3e^{y} - B_3e^{-y} - C_3(2-4\mu_3-y)e^{y} - D_3(2-4\mu_3+y)e^{-y}\right]$$

$$u:\quad A_2e^{y} + B_2e^{-y} + C_2(1+y)e^{y} - D_2(1-y)e^{-y} =$$
$$K_u\left[A_3e^{y} + B_3e^{-y} + C_3(1+y)e^{y} - D_3(1-y)e^{-y}\right]$$

Boundary conditions at the third interface ($z = H_1 + H_2 + H_3$):

$$\sigma_z:\quad A_3e^{z} + B_3e^{-z} - C_3(1-2\mu_3-z)e^{z} + D_3(1-2\mu_3+z)e^{-z} =$$
$$A_4e^{z} + B_4e^{-z} - C_4(1-2\mu_4-z)e^{z} + D_4(1-2\mu_4+z)e^{-z}$$

$$\tau_{rz}:\quad A_3e^{z} - B_3e^{-z} + C_3(2\mu_3+z)e^{z} + D_3(2\mu_3-z)e^{-z} =$$
$$A_4e^{z} - B_4e^{-z} + C_4(2\mu_4+z)e^{z} + D_4(2\mu_4-z)e^{-z}$$

3.2. Algebraical analysis of an isotropic four-layer system with fixed bottom.

3.2.1. Boundary conditions.

In this analysis, we choose as fixed bottom condition the one described in § 1.2.2. (w = 0).

We write

$$A_i = A_i m^2 \qquad B_i = B_i m^2 \qquad C_i = C_i m \qquad = D_i m$$

$$F_w = \frac{E_1(1+\mu_2)}{E_2(1+\mu_1)} \qquad F_n = k_1 \frac{E_1(1+\mu_2)}{E_2(1+\mu_1)}$$

$$K_w = \frac{E_2(1+\mu_3)}{E_3(1+\mu_2)} \qquad K_n = k_2 \frac{E_2(1+\mu_3)}{E_3(1+\mu_2)}$$

$$L_w = \frac{E_3(1+\mu_4)}{E_4(1+\mu_3)} \qquad L_n = k_3 \frac{E_3(1+\mu_4)}{E_4(1+\mu_3)}$$

$$x = mH_1$$
$$y = m(H_1 + H_2)$$
$$z = m(H_1 + H_2 + H_3)$$
$$t = m(H_1 + H_2 + H_3 + H_4)$$
$$n = -2t + z$$

where H_1, H_2, H_3 and H_4 are the thicknesses of the four layers.

Boundary nditions at the surface (z = 0):

$$\sigma_z = p \qquad A_1 + B_1 - C_1(1-2\mu_1) + D_1[1-2\mu_1) = 1$$

$$\tau_{rz} = 0 \qquad A_1 - B_1 + C_1 \cdot 2\mu_1 + D_1 2\mu_1 = 0$$

```
E1=100000        U1=0.50        H1=10.00        A= 10.00        P= 1.00
E2= 10000        U2=0.50        H2=20.00        R=  0.00
E3=  1000        U3=0.50
```

	SURFACE	BASE COUCHE 1	SURFACE COUCHE 2	BASE COUCHE 2	SURFACE COUCHE 3
CONTRAINTES VERTICALES		+0.247002	+0.247002	+0.029472	+0.029472
CONTRAINTES RADIALES		-1.864850	+0.035817	-0.204340	+0.006091
CONTRAINTES CIRCONFER.		-1.864850		-0.204340	+0.006091
FLECHES	+2.183E-03				

Table 1. Stresses in a three-layer system (BASIC Program).

RESULTATS DANS L'AXE DE LA CHARGE.

```
E1=100000.     U1=0.50      H1=10.00      A= 10.00      P= 1.00
E2= 10000.     U2=0.50      H2=20.00      R=   .00
E3=  1000.     U3=0.50
```

	SURFACE	BASE COUCHE1	SURFACE COUCHE2	BASE COUCHE2	SURFACE COUCHE3
CONTRAINTES VERTICALES	*	.247002	.247002	.029472	.029472
CONTRAINTES RADIALES	*	-1.864850	.035817	-.204340	.006091
CONTRAINTES CIRCONF.	*	-1.864850	.035817	-.204340	.006091
FLECHES	.0022	*	*	*	*

Table 2. Stresses in a three-layer system (FORTRAN 77 Program).

3.1.4. Comparison with existing programs.

First we have computed the stresses in a three-layer isotropic system with a program written in BASIC and for which the equations had been developped in complete closeform (VAN CAUWELAERT, 1983).
The accuracy of this program had been checked earlier with the results published by JONES (1962).
Then we have written a new program in FORTRAN 77 based on the developments presented in paragraphs 3.1.1. and 3.1.2. and calculated with this program the stresses in the same three-layer system.
The results obtained by the program in BASIC are given in table 1, those obtained by the program in FORTRAN 77 are given in table 2: we notice that the agreement between both is perfect.
The complete listing of the FORTRAN 77 program is given in appendix.

The matrices are dimensionned and upbuilt in the beginning of the program. The variables x (=mH_1) and y (=mH_1 + mH_2) are introduced in the matrices by instruction 214. The products between matrices are carried out in the same instruction 214 by calling subroutines 800, 810, 830 and 860.

3.1.3. Relation for the vertical deflection at the surface.

We notice that the values of the parameters given by (10) and (11) all converge normally, except for parameters B_1 and D_1 which contain each a constant in the numerator:

$$\frac{1}{4}(1+k)(1+L)\,B_3 = \frac{1}{4}(1+k)(1+L)\,\frac{4(1+k)(1+L) + 4d_{22}}{\nabla}$$

$$= \frac{[(1+k)(1+L)]^2}{\nabla} + \frac{(1+k)(1+L).\,d_{22}}{\nabla}$$

The influence of this constant can be eliminated as indicated in § 2.3.

Here it is nevertheless easier to eliminate B_1.

From the boundary conditions at the surface, we have

$$B_1 = 1 - A_1$$

The relation for the deflection at the surface is given by

$$w = -pa\int_0^\infty \frac{J_1(ma)}{m}.\ \frac{1+\mu_1}{E_1}\left[A_1 - B_1\right] dm$$

$$= -pa.\frac{1+\mu_1}{E_1}\int_0^\infty \frac{J_1(ma)}{m}\,(1-2A_1)\,dm$$

$$= -pa\,\frac{1+\mu_1}{E_1}\int_0^\infty \frac{J_1(ma)}{m}dm + 2pa\,\frac{1+\mu_1}{E_1}\int_0^\infty \frac{J_1(ma)}{m}A_1\,dm \qquad (12)$$

$$= -pa\,\frac{1+\mu_1}{E_1} + 2pa\,\frac{1+\mu_1}{E_1}\int_0^\infty \frac{J_1(ma)}{m}A_1\,dm$$

The first integral of (12) is solved analytically, the second one converges fast and safely during the numerical integration procedure.

The parameters A_1, B_1, C_1 and D_1 are then obtained from (6)

$$
\begin{aligned}
A_1 &= -\frac{1}{4}\Big\{(1+k)e^{-2y}\left[M_{3141}(1,1)\,B_3 + M_{3141}(1,2)\,D_3\right] \\
&\qquad + (1+L)e^{-2x}\left[M_{1122L}(1,1)\,B_3 + M_{1122L}(1,2)\,D_3\right]\Big\} \\
B_1 &= \frac{1}{4}\Big\{(1+k)(1+L)\,B_3 + e^{-2(y-x)}\left[M_{1241}(2,1)\,B_3 + M_{1241}(2,2)\,D_3\right]\Big\} \\
C_1 &= -\frac{1}{4}\Big\{(1+k)e^{-2y}\left[M_{3141}(3,1)\,B_3 + M_{3141}(3,2)\,D_3\right] \\
&\qquad + (1+L)e^{-2x}\left[M_{1122L}(3,1)\,B_3 + M_{1122L}(3,2)\,D_3\right]\Big\} \\
D_1 &= \frac{1}{4}\Big\{(1+k)(1+L)\,D_3 + e^{-2(y-x)}\left[M_{1241}(4,1)\,B_3 + M_{1241}(4,2)\,D_3\right]\Big\}
\end{aligned}
\tag{10}
$$

Parameters A_2, B_2, C_2 and D_2 are obtained from (3)

$$
\begin{aligned}
A_2 &= -\frac{1}{2}e^{-2y}\left[M_{3141}(1,1)\cdot B_3 + M_{3141}(1,2)\cdot D_3\right] \\
B_2 &= \frac{1}{2}(1+L)\,B_3 \\
C_2 &= -\frac{1}{2}e^{-2y}\left[M_{3141}(3,1)\cdot B_3 + M_{3141}(3,2)\,D_3\right] \\
D_2 &= \frac{1}{2}(1+L)\,D_3
\end{aligned}
\tag{11}
$$

$$\mathrm{I}.\,M_{3141} = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}$$

$$\mathrm{I}.\,M_{1241} = \begin{pmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{pmatrix}$$

$$\mathrm{I}.\,M_{1122L} = \begin{pmatrix} c_{11} & c_{12} \\ c_{21} & c_{22} \end{pmatrix}$$

The terms a_{ij}, b_{ij} and c_{ij} are the results of very simple computerprocedures.
We can now write (8) in extended form:

$$\left[(1+k)(1+L) - (1+k).a_{11}.e^{-2y} + b_{11}.e^{-2(y-x)} - (1+L).c_{11}.e^{-2x}\right] B_3$$
$$+\left[-(1+k).a_{12}.e^{-2y} + b_{12}.e^{-2(y-x)} - (1+L).c_{12}.e^{-2x}\right] D_3 = 4$$

$$\left[-(1+k)(1+L) - (1+k).a_{21}.e^{-2y} + b_{21}.e^{-2(y-x)} - (1+L).c_{21}.e^{-2x}\right] B_3$$
$$+\left[(1+k)(1+L) - (1+k).a_{22}.e^{-2y} + b_{22}.e^{-2(y-x)} - (1+L).c_{22}.e^{-2x}\right] D_3 = 0$$

We write this

$$\left[(1+k)(1+L) + d_{11}\right].B_3 + d_{12}.D_3 = 4$$
$$\left[-(1+k)(1+L) + d_{21}\right] B_3 + \left[(1+k)(1+L) + d_{22}\right] D_3 = 0$$

where $(1 + K).(1 + L)$ is, as a matter of fact, a constant and all d_{11}, d_{12}, d_{21} and d_{22} converge normally.

Finally one obtains

$$B_3 = \frac{4(1+k)(1+L) + 4d_{22}}{\nabla} \qquad D_3 = \frac{4(1+k)(1+L) - 4d}{\nabla}$$

where

$$\nabla = \left[(1+k)(1+L)^2 + (1+k)(1+L)(d_{11} + d_{12} + d_{22}) + d_{11}.d_{22} - d_{12}.d_{21}\right]$$

$$M_{1122} \cdot M_{3141} = 0$$

$$M_{1221} \cdot M_{3141} = \begin{pmatrix} 0 & 0 \\ + & + \\ 0 & 0 \\ + & + \end{pmatrix} = M_{1241}$$

$$M_{1122} \cdot U_L = \begin{pmatrix} + & + \\ 0 & 0 \\ + & + \\ 0 & 0 \end{pmatrix} = M_{1122L}$$

$$M_{1221} \cdot U_L = 0$$

so that

$$(A_1 B_1 C_1 D_1)^T = \frac{1}{4}\Big[(1+k)(1+L) \cdot U_L - (1+k)e^{-2y} M_{3141} + e^{-2(y-x)} M_{1241} - (1+L)e^{-2x} M_{1122L}\Big](B_3 \quad D_3)^T \tag{6}$$

The positive exponent e^{2x} has again disappeared together with the highest negative exponent $e^{-2(y+x)}$.

Finally we consider the conditions at the surface

$$I(A_1 B_1 C_1 D_1)^T = (1 \quad 0)^T \tag{7}$$

We replace in (7), $(A_1\ B_1\ C_1\ D_1)^T$ by its value from (6)

$$\frac{1}{4} I \Big[(1+k)(1+L) \cdot U_L - (1+k)e^{-2y} \cdot M_{3141} + e^{-2(y-x)} \cdot M_{1241} - (1+L)e^{-2x} \cdot M_{1122L}\Big](B_3 \quad D_3)^T = (1 \quad 0)^T \tag{8}$$

$$(1+k)(1+L) I \cdot U_L = (1+k)(1+L)\begin{pmatrix} 1 & 0 \\ -1 & 1 \end{pmatrix}$$

$$M_{11}.M_{21} + M_{12}.M_{22} = -(1+k)\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} = -(1+k)U_k$$

$$M_{11}.M_{22} = \begin{pmatrix} 0 & + & 0 & + \\ 0 & 0 & 0 & 0 \\ 0 & + & 0 & + \\ 0 & 0 & 0 & 0 \end{pmatrix} = M_{1122}$$

$$M_{12}.M_{21} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ + & 0 & + & 0 \\ 0 & 0 & 0 & 0 \\ + & 0 & + & 0 \end{pmatrix} = M_{1221}$$

We replace in (5), $(A_2\ B_2\ C_2\ D_2)^T$ by its value from (3)

$$(A_1 B_1 C_1 D_1)^T = \frac{1}{4}\left[-(1+k).U_k + e^{-2x}.M_{1122} + e^{2x}M_{1221}\right].$$
$$\left[e^{-2y}M_{3141} - (1+L)U_L\right](B_3 D_3)^T$$

$$= \frac{1}{4}\Big[(1+k)(1+L)U_k.U_L - (1+k)e^{-2y}.U_k.M_{3141}$$
$$+ e^{-2(y+x)}.M_{1122}.M_{3141} + e^{-2(y-x)}.M_{1221}.M_{3141}$$
$$-(1+L)e^{-2x}.M_{1122}.U_L - (1+L)e^{2x}.M_{1221}.U_L\Big](B_3\ D_3)^T$$

$$U_k.U_L = \begin{pmatrix} 0 & 0 \\ 1 & 0 \\ 0 & 0 \\ 0 & 1 \end{pmatrix} = U_L$$

$$U_k.M_{3141} = \begin{pmatrix} + & + \\ 0 & 0 \\ + & + \\ 0 & 0 \end{pmatrix} = M_{3141}$$

The product we are interested in for later computation is

$$(M_{31} \cdot M_{41} + M_{32} \cdot M_{42}) \cdot (M_{52} \cdot M_{62}) =$$

$$\begin{pmatrix} 0 & 0 \\ k_1 L_1 & k_1 L_3 + k_3 L_2 \\ 0 & 0 \\ 0 & k_2 L_2 \end{pmatrix} + y(k_1 - k_2) \begin{pmatrix} 0 & 0 \\ 0 & L_2 \\ 0 & 0 \\ 0 & 0 \end{pmatrix}$$

$$+ z(L_1 - L_2) \begin{pmatrix} 0 & 0 \\ 0 & k_1 \\ 0 & 0 \\ 0 & 0 \end{pmatrix} + yz(k_1 - k_2)(L_1 - L_2) \begin{pmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{pmatrix} = N_{11}$$

This product contains a constant, a linear function of y and a linear function of z. The term, function of y.z, a nonlinaer function, disappears.

The other products are

$$M_{3142} \cdot M_{5261} = \begin{pmatrix} + & + \\ 0 & 0 \\ + & + \\ 0 & 0 \end{pmatrix} = N_{21}$$

$$M_{3142} \cdot M_{52} \cdot M_{62} = \begin{pmatrix} + & + \\ 0 & 0 \\ + & + \\ 0 & 0 \end{pmatrix} = N_{22}$$

$$M_{3241} \cdot M_{5161} = \begin{pmatrix} 0 & 0 \\ + & + \\ 0 & 0 \\ + & + \end{pmatrix} = N_{12}$$

$$M_{3241} \cdot M_{5162} = \begin{pmatrix} 0 & 0 \\ + & + \\ 0 & 0 \\ + & + \end{pmatrix} = N_{13}$$

$$\left(M_{31} \cdot M_{41} + M_{32} \cdot M_{42}\right) \cdot M_{5161} = \begin{pmatrix} + & + \\ 0 & 0 \\ + & + \\ 0 & 0 \end{pmatrix} = N_{23}$$

$$\left(M_{31} \cdot M_{41} + M_{32} \cdot M_{42}\right) \cdot M_{5162} = \begin{pmatrix} + & + \\ 0 & 0 \\ + & + \\ 0 & 0 \end{pmatrix} = N_{24}$$

$$\left(M_{31} \cdot M_{41} + M_{32} \cdot M_{42}\right) \cdot M_{5261} = \begin{pmatrix} 0 & 0 \\ + & + \\ 0 & 0 \\ + & + \end{pmatrix} = N_{14}$$

with

$$N_{1i} = \begin{pmatrix} 0 & 0 \\ + & + \\ 0 & 0 \\ + & + \end{pmatrix} \qquad N_{2i} = \begin{pmatrix} + & + \\ 0 & 0 \\ + & + \\ 0 & 0 \end{pmatrix}$$

The final relation becomes

$$\left(A_2\, B_2\, C_2\, D_2\right)^T = \frac{1}{16\,(1-\mu_2)(1-\mu_3)} \cdot$$

$$\Big[e^{-2y} \cdot e^{-2(t-z)} \cdot N_{21} + e^{-2y} \cdot N_{22} + e^{-2(t-y)} N_{12}$$

$$+ e^{-2(z-y)} \cdot N_{13} + e^{-2t} \cdot N_{23} + e^{-2z} \cdot N_{24}$$

$$+ e^{-2(t-z)} N_{14} + N_{11} \Big] \left(B_4 \;\; D_4\right)^T \qquad (801.2)$$

Next we write the conditions at the first interface in matrixform

$$(A_1\ B_1\ C_1\ D_1)^T = M_1^{-1} \cdot M_2\, (A_2\ B_2\ C_2\ D_2)^T$$

$$M_1^{-1} = -\frac{1}{4(1-\mu_1)}\, e^{-x} \begin{pmatrix} -(1+x) & -(2-4\mu_1-x) & -(2\mu_1+x) & -(1-2\mu_1-x) \\ 0 & 0 & 0 & 0 \\ 1 & -1 & 1 & -1 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

$$-\frac{1}{4(1-\mu_1)}\, e^{x} \begin{pmatrix} 0 & 0 & 0 & 0 \\ -(1-x) & (2-4\mu_1+x) & (2\mu_1-x) & -(1-2\mu_1+x) \\ 0 & 0 & 0 & 0 \\ -1 & -1 & 1 & 1 \end{pmatrix}$$

$$M_2 = e^{x} \begin{pmatrix} 1 & 0 & -(1-2\mu_2-x) & 0 \\ 1 & 0 & (2\mu_2+x) & 0 \\ F & 0 & -F(2-4\mu_2-x) & 0 \\ F & 0 & F(1+x) & 0 \end{pmatrix}$$

$$+\, e^{-x} \begin{pmatrix} 0 & 1 & 0 & (1-2\mu_2+x) \\ 0 & -1 & 0 & (2\mu_2-x) \\ 0 & -F & 0 & -F(2-4\mu_2+x) \\ 0 & F & 0 & -F(1-x) \end{pmatrix}$$

$$(A_1, B_1, C_1, D_1)^T = -\frac{1}{4(1-\mu_1)} \left[e^{-x} M_{11} + e^{x} M_{12} \right] \cdot \left[e^{x} M_{21} + e^{-x} M_{22} \right] (A_2 B_2 C_2 D_2)^T$$

$$= -\frac{1}{4(1-\mu_1)} \left[M_{11} \cdot M_{21} + e^{-2x} M_{11} \cdot M_{22} + e^{2x} M_{12} \cdot M_{21} + M_{12} M_{22} \right] (A_2 B_2 C_2 D_2)^T$$

$$M_{11} \cdot M_{21} + M_{12} \cdot M_{22} =$$

$$\begin{pmatrix} F_1 & 0 & -F_3 & 0 \\ 0 & F_1 & 0 & F_3 \\ 0 & 0 & F_2 & 0 \\ 0 & 0 & 0 & F_2 \end{pmatrix} + x(F_1 - F_2) \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

The first matrix is a constant, the second is a linear function of x.

$$M_{11} \cdot M_{22} = \begin{pmatrix} 0 & + & 0 & + \\ 0 & 0 & 0 & 0 \\ 0 & + & 0 & + \\ 0 & 0 & 0 & 0 \end{pmatrix} = M_{1122}$$

$$M_{12} \cdot M_{21} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ + & 0 & + & 0 \\ 0 & 0 & 0 & 0 \\ + & 0 & + & 0 \end{pmatrix} = M_{1221}$$

$$(A_1 B_1 C_1 D_1)^T = -\frac{1}{64(1-\mu_1)(1-\mu_2)(1-\mu_3)} \cdot \left[M_{11} \cdot M_{21} + M_{12} \cdot M_{22} + M_{1122} e^{-2x} + M_{1221} e^{2x} \right] \cdot$$

$$\Big[e^{-2y} \cdot e^{-2(r-z)} \cdot N_{21} + e^{-2y} \cdot N_{22} + e^{-2(r-y)} N_{12}$$
$$+ e^{-2(z-y)} \cdot N_{13} + e^{-2r} \cdot N_{23} + e^{-2z} \cdot N_{24}$$
$$+ e^{-2(r-z)} \cdot N_{14} + N_{11} \Big] \left(B_4 \quad D_4 \right)^T$$

The products which will not converge normally are

$$e^{2x} \cdot e^{-2(r-y)} M_{1221} \cdot M_{3241} \cdot M_{5161}$$

$$e^{2x} e^{-2(z-y)} M_{1221} \cdot M_{3241} \cdot M_{5162}$$

$$e^{2x} e^{-2(r-z)} M_{1221} \left(M_{31} \cdot M_{41} + M_{32} \cdot M_{42} \right) \cdot M_{5261}$$

because that it can happen that $\quad x > r-y$

or $\quad x > z-y$

or $\quad x > r-z$

and

$$e^{2x} \cdot M_{1221} \left(M_{31} \cdot M_{41} + M_{32} \cdot M_{42} \right) \cdot \left(M_{52} \cdot M_{62} \right)$$

One easily verifies that

$$M_{1221} \cdot M_{3241} = 0$$

$$M_{1221} \cdot \left(M_{31} \cdot M_{41} + M_{32} \cdot M_{42} \right) \cdot M_{5261} = 0$$

$$M_{1221} \cdot \left(M_{31} \cdot M_{41} + M_{32} \cdot M_{42} \right) \cdot M_{52} \cdot M_{62} = 0$$

so that again the concerned products disappear from the relation.

The product, we are interested in for later computation, is

$$\left[M_{11} \cdot M_{21} + M_{12} \cdot M_{22} \right] \left[M_{31} \cdot M_{41} + M_{32} \cdot M_{42} \right] \left[M_{52} \cdot M_{62} \right] =$$

$$\begin{pmatrix} 0 & 0 \\ F_1 k_1 L_1 & F_1 k_1 L_3 + F_1 k_3 L_2 + F_3 k_2 L_2 \\ 0 & 0 \\ 0 & F_2 k_2 L_2 \end{pmatrix} + y(k_1 - k_2) \begin{pmatrix} 0 & 0 \\ 0 & F_1 L_2 \\ 0 & 0 \\ 0 & 0 \end{pmatrix}$$

$$+ x(F_1 - F_2) \begin{pmatrix} 0 & 0 \\ 0 & k_2 L_2 \\ 0 & 0 \\ 0 & 0 \end{pmatrix} + z(L_1 - L_2) \begin{pmatrix} 0 & 0 \\ 0 & F_1 k_1 \\ 0 & 0 \\ 0 & 0 \end{pmatrix} = T_{11}$$

One has $\quad F_1 k_1 L_3 + F_1 k_3 L_2 + F_3 k_2 L_2 = \frac{1}{2} \left[k_1 L_1 F_1 (1 - 4\mu_4) - k_2 L_2 F_2 (1 - 4\mu_1) \right]$

One verifies that the products

$$M_{1122} \cdot N_i = 0$$

$$M_{1221} \cdot N_i = 0$$

The other products are

$$\left(M_{11} \cdot M_{21} + M_{12} \cdot M_{22}\right) \cdot N_{1i} = \begin{pmatrix} 0 & 0 \\ + & + \\ 0 & 0 \\ + & + \end{pmatrix} = T_{1i}$$

$$\left(M_{11} \cdot M_{21} + M_{12} \cdot M_{22}\right) \cdot N_{2i} = \begin{pmatrix} + & + \\ 0 & 0 \\ + & + \\ 0 & 0 \end{pmatrix} = T_{2i}$$

$$M_{1122} \cdot N_{1i} = \begin{pmatrix} + & + \\ 0 & 0 \\ + & + \\ 0 & 0 \end{pmatrix} = T_{3i} \qquad M_{1221} \cdot N_{2i} = \begin{pmatrix} 0 & 0 \\ + & + \\ 0 & 0 \\ + & + \end{pmatrix} = T_{4i}$$

and finally

$$\begin{aligned}
(A_1 B_1 C_1 D_1)^T = -\frac{1}{64(1-\mu_1)(1-\mu_2)(1-\mu_3)} \cdot \\
\Big[e^{-2y} \cdot e^{-2(t-z)} \cdot T_{21} + e^{-2y} \cdot T_{22} + e^{-2(t-y)} \cdot T_{12} + e^{-2(z-y)} \cdot T_{13} \\
+ e^{-2t} \cdot T_{23} + e^{-2z} \cdot T_{24} + e^{-2(t-z)} \cdot T_{14} + e^{-2x} \cdot e^{-2(t-y)} \cdot T_{32} \\
+ e^{-2x} \cdot e^{-2(z-y)} \cdot T_{33} + e^{-2x} \cdot e^{-2(t-z)} \cdot T_{34} + e^{-2x} \cdot T_{31} \\
+ e^{-2(y-x)} \cdot e^{-2(t-z)} \cdot T_{41} + e^{-2(y-x)} \cdot T_{42} + e^{-2(t-x)} \cdot T_{43} \\
+ e^{-2(z-x)} \cdot T_{44} + T_{11} \Big] (B_4 \;\; D_4)^T
\end{aligned} \qquad (A1.3)$$

Finally we considerthe boundary conditio s at the surface:

$$\begin{pmatrix} 1 & 1 & -(1-2\mu_1) & (1-2\mu_1) \\ 1 & -1 & 2\mu_1 & 2\mu_1 \end{pmatrix} \cdot (A_1 B_1 C_1 D_1)^T = (1 \;\; 0)^T$$

that we can transform into

$$-\frac{1}{64(1-\mu_1)(1-\mu_2)(1-\mu_3)} \cdot \begin{pmatrix} 1 & 1 & -(1-2\mu_1) & (1-2\mu_1) \\ 1 & -1 & 2\mu_1 & 2\mu_1 \end{pmatrix} \cdot$$

$$\left[\begin{pmatrix} c_{11} & c_{12} \\ c_{21} & c_{22} \\ c_{31} & c_{32} \\ c_{41} & c_{42} \end{pmatrix} + \begin{pmatrix} 0 & 0 \\ d_{21} & d_{22} \\ 0 & 0 \\ 0 & d_{42} \end{pmatrix} \right] \cdot (B_4 \;\; D_4)^T = (1 \;\; 0)^T$$

Extending matrix (d_{ij}), equal to T_{11}, we obtain

$$-\frac{1}{64(1-\mu_1)(1-\mu_2)(1-\mu_3)} \left\{ \left[\begin{pmatrix} 1 & 1 & -(1-2\mu_1) & (1-2\mu_1) \\ 1 & -1 & 2\mu_1 & 2\mu_1 \end{pmatrix} \begin{pmatrix} c_{11} & c_{12} \\ c_{21} & c_{22} \\ c_{31} & c_{32} \\ c_{41} & c_{42} \end{pmatrix} \right. \right.$$

$$+ \begin{pmatrix} F_1 k_1 L_1 & \frac{1}{2}\left[F_1 k_1 L_1 (1-4\mu_4) + F_2 k_2 L_2\right] \\ -F_1 k_1 L_1 & -\frac{1}{2}\left[F_1 k_1 L_1 (1-4\mu_4) - F_2 k_2 L_2\right] \end{pmatrix}$$

$$\left. \left. + \left[x(F_1 - F_2) \cdot k_2 L_2 + y(k_1 - k_2) F_1 L_2 + z(L_1 - L_2) F_1 k_1\right] \begin{pmatrix} 0 & 1 \\ 0 & -1 \end{pmatrix} \right\} (B_4 D_4)^T = (1 \;\; 0)^T$$

This last relation can be writtenin extended form

$$\left[c_{11} + c_{21} - (1-2\mu_1)c_{31} + (1-2\mu_1)c_{41} + F_1k_1L_1\right]B_4$$

$$+\Big\{c_{12} + c_{22} - (1-2\mu_1)c_{32} + (1-2\mu_1)c_{42} + \frac{1}{2}\left[F_1k_1L_1(1-4\mu_4) + F_2k_2L_2\right]$$

$$+x(F_1-F_2)k_2L_2 + y(k_1-k_2)F_1L_2 + z(L_1-L_2)F_1k_1\Big\} = -\frac{(1-\mu_1)(1-\mu_2)(1-\mu_3)}{64}$$

$$\left[c_{11} - c_{21} + 2\mu_1c_{31} + 2\mu_4c_{41} - F_1k_1L_1\right]B_4$$

$$+\Big\{c_{12} - c_{22} + 2\mu_1c_{32} + 2\mu_1c_{42} - \frac{1}{2}\left[F_1k_1L_1(1-4\mu_4) - F_2k_2L_2\right]$$

$$-x(F_1-F_2)k_2L_2 - y(k_1-k_2)F_1L_2 - z(L_1-L_2)F_1k_1\Big\} = 0$$

We can write this system as follows

$$(n_{11} + F_1k_1L_1)B_4 + \Big\{n_{12} + \frac{1}{2}\left[F_1k_1L_1(1-4\mu_4) + F_2k_2L_2\right] + f(xyz)\Big\}D_4 =$$

$$= -\frac{(1-\mu_1)(1-\mu_2)(1-\mu_3)}{64}$$

$$(n_{21} - F_1k_1L_1)B_4 + \Big\{n_{22} - \frac{1}{2}\left[F_1k_1L_1(1-4\mu_4) \quad F_2k_2L_2\right] - f(xyz)\Big\}D_4 = 0$$

where all n_{ij} converge absolutely normally and f(xyz) is a linear function of x, y and z.

Solving this system, we obtain

$$B_4 = -\frac{(1-\mu_1)(1-\mu_2)(1-\mu_3)}{64}\Big\{n_{22} - \frac{1}{2}\left[F_1k_1L_1(1-4\mu_4) - F_2k_2L_2\right] - f(xyz)\Big\}\cdot\frac{1}{\nabla}$$

$$D_4 = \frac{(1-\mu_1)(1-\mu_2)(1-\mu_3)}{64}\left[n_{21} - F_1k_1L_1\right]\cdot\frac{1}{\nabla}$$

where

$$\nabla = n_{11} \cdot n_{22} - n_{12} \cdot n_{21} + F_1 k_1 L_1 \left(n_{12} + n_{22} \right) - f(xyz) \left(n_{11} + n_{21} \right)$$
$$- n_{11} \frac{1}{2} \left[F_1 k_1 L_1 \left(1 - 4\mu_4 \right) - F_2 k_2 L_2 \right] - n_{21} \frac{1}{2} \left[F_1 k_1 L_1 \left(1 - 4\mu_4 \right) + F_2 k_2 L_2 \right]$$
$$- F_1 k_1 L_1 \cdot \frac{1}{2} \left[F_1 k_1 L \left(1 - 4\mu_4 \right) - F_2 k_2 L_2 \right] + F_1 k_1 L_1 \frac{1}{2} \left[F_1 k_1 L_1 \left(1 - 4\mu_2 \right) + F_2 k_2 L_2 \right]$$
$$+ F_1 k_1 L_1 \, f(xyz) - F_1 k_1 L_1 \, f(xyz)$$

$$\nabla = f \left(n_{11}, n_{12}, n_{21}, n_{22} \right) + F_1 k_1 L_1 \cdot F_2 k_2 L_2$$

The linear function f(xyz) has disappeared in the denominator.
The function $f(n_{ij})$ converges normally so that the limit value for the denominator is a constant:

$$\lim_{m \to \infty} \nabla = F_1 k_1 L_1 \cdot F_2 k_2 L_2$$

3.2.3. Values of the parameters A_i, D_i.

At the bottom of the first layer, parameters A_1 and C_1 are factors of the positive exponent e^x. Parameters B_1 and D_1 are factors of the negative exponent e^{-x}.
Thus when computing stresses in the first layer we have to input at least two positive exponents which necessarly will lead to overflowproblems.
Therefore we express in the program next modified values of the parameters A_1, B_1, C_1 and D_1: $A_1.e^x$, B_1, $C_1.e^x$, D_1.
Then if we have to compute a stress, let us say at a depth 2x/3, we multiply the parameters $A_1.e^x$ and $C_1.e^x$ by the negative exponent $e^{-x/3}$ and the parameters B_1 and D_1 by the negative exponent $e^{-2x/3}$ and avoid, in doing so, all overflow problems.

The values of the parameters A_1, B_1, C_1 and D_1 are given by (sol. 3) of §3.2.2.
To obtain the values for $A_1.e^x$, B_1, $C_1.e^x$ and D_1 we split (sol. 3) into two parts:

$$(A_1 \quad 0 \quad C_1 \quad 0)^T + (0 \quad B_1 \quad 0 \quad D_1)^T =$$

$$-\frac{1}{64(1-\mu_1)(1-\mu_2)(1-\mu_3)} \left[\qquad \qquad \right] (B_4 \quad D_4)^T$$

We notice now that the matrices T_{1i} and T_{4i} contain nothing but zeros in their first and third rows, so that

$$T_{1i} \cdot (B_4 \quad D_4)^T = \begin{pmatrix} 0 & 0 \\ + & + \\ 0 & 0 \\ + & + \end{pmatrix} (B_4 \quad D_4)^T = \begin{pmatrix} 0 \\ + \\ 0 \\ + \end{pmatrix}$$

and that the matrices T_{2i} and T_{3i} contain nothing but zeros in their second and fourth rows, so that

$$T_{2i} \cdot (B_4 \quad D_4)^T = \begin{pmatrix} + & + \\ 0 & 0 \\ + & + \\ 0 & 0 \end{pmatrix} (B_4 \quad D_4)^T = \begin{pmatrix} + \\ 0 \\ + \\ 0 \end{pmatrix}$$

Thus the values of A_1 and C_1 depend only on the matrices T_{2i} and T_{3i}. This has to be so, because only the exponents multiplying the matrices T_{2i} and T_{3i} do not overflow when they are multiplied by e^x.
The exponents multiplying the matrices T_{1i} and T_{4i} could overflow when multiplied by e^x: for example, the exponent $e^{-2(t - z)}$ multiplying the matrix T_{14} if $x > 2(t - z)$.
We obtain then

$$e^x \cdot (A_1 \quad 0 \quad C_1 \quad 0)^T = - \frac{1}{64(1-\mu_1)(1-\mu_2)(1-\mu_3)} \cdot$$

$$\Big[e^{-(2y-x)} \cdot e^{-2(t-z)} \cdot T_{21} + e^{-(2y-x)} \cdot T_{22} + e^{-(2t-x)} \cdot T_{23} + e^{-(2z-x)} \cdot T_{24} + e^{-x} \cdot e^{-2(t-y)} \cdot T_{32} + e^{-x} \cdot e^{-2(z-y)} \cdot T_{33} + e^{-x} \cdot e^{-2(t-z)} \cdot T_{34} + e^{-x} \cdot T_{31} \Big] (B_4 \quad D_4)^T$$

$$(0 \quad B_1 \quad 0 \quad D_1)^T = - \frac{1}{(1-\mu_1)(1-\mu_2)(1-\mu_3)} \cdot$$

$$\Big[e^{-2(t-y)} \cdot T_{12} + e^{-2(z-y)} \cdot T_{1} + e^{-2(t-z)} \cdot T_{14} + e^{-2(y-x)} \cdot e^{-2(t-z)} \cdot T_{41} + e^{-2(y-x)} \cdot T_{42} + e^{-2(t-x)} \cdot T_{43} + e^{-2(z-x)} T_{44} + T_{11} \Big] (B_4 \quad D_4)^T$$

We follow the same procedure for the parameters A_2, B_2, C_2 and D_2 and obtain from (sol. 2):

$$e^y (A_2 \quad 0 \quad C_2 \quad 0)^T = \frac{1}{16(1-\mu_2)(1-\mu_3)} \cdot$$

$$\Big[e^{-y} \cdot e^{-2(t-z)} \cdot N_{21} + e^{-y} \cdot N_{22} + e^{-(2t-y)} \cdot N_{23} + e^{-(2z-y)} \cdot N_{24} \Big] (B_4 \quad D_4)^T$$

$$(0 \quad B_2 \quad 0 \quad D_2)^T = \frac{1}{16(1-\mu_2)(1-\mu_3)}$$

$$\Big[e^{-2(t-y)} \cdot N_{12} + e^{-2(z-y)} \cdot N_{13} + e^{-2(t-z)} N_{14} + N_{11} \Big] (B_4 \quad D_4)^T$$

Finally we obtain for the parameters A_3, B_3, C_3 and D_3(sol. 1):

$$e^{z}\begin{pmatrix}A_3 & 0 & C_3 & 0\end{pmatrix}^T = -\frac{1}{4(1-\mu_3)}\cdot\left[e^{-(2t-z)}\cdot M_{5161} + e^{-z}\cdot M_{5162}\right]\begin{pmatrix}B_4 & D_4\end{pmatrix}^T$$

$$\begin{pmatrix}0 & B_3 & 0 & D_3\end{pmatrix}^T = -\frac{1}{4(1-\mu_3)}\cdot\left[e^{-2(t-z)}\cdot M_{5261} + M_{52}\cdot M_{62}\right]\begin{pmatrix}B_4 & D_4\end{pmatrix}^T$$

3.2.4. The deflection at the surface.

The vertical deflection at the surface is given by

$$W = -pa\int_0^\infty \frac{J_0(mr).J_1(ma)}{m}\,\frac{1+\mu_1}{E_1}\left[A_1 - B_1 - 2C_1(1-2\mu_1) - 2D_1(1-2\mu_1)\right]dm$$

The parameters B_1 and D_1 do not converge normally because they contain a constant in their numerator.

Utilizing the boundary conditions at the surface, we express then B_1 and D_1 in function of A_1 and C_1:

$$B_1 = 2\mu_1 + A_1(1-4\mu_1) + 4\mu_1 C_1(1-2\mu_1)$$
$$D_1 = 1 - 2A_1 + C_1(1-4\mu_1)$$

so that

$$A_1 - B_1 - 2C_1(1-2\mu_1) - 2D_1(1-2\mu_1) =$$
$$-2(1-\mu_1)\left[1 - 2A_1 + 2C_1(1-2\mu_1)\right]$$

and

$$W = \frac{2(1-\mu_1^2)}{E_1}\cdot pa\int_0^\infty \frac{J_0(mr).J_1(ma)}{m}\cdot dm$$
$$-\frac{4(1-\mu_1^2)}{E_1}\cdot pa\int_0^\infty \frac{J_0(mr).J_1(ma)}{m}\left[A_1 - (1-2\mu_1)C_1\right]dm$$

The first integral can be solved analytically (§ 2.3). The second one converges absolutely normally.

3.2.5. The stresses and displacements in the first layer.

We compute, for example, the vertical stress at a depth $0 < H < H_1$.
Its value is given by

$$\sigma_{z,H} = pa \int_0^\infty J_0(mr) \cdot J_1(ma) \left[A_1 \cdot e^x \cdot e^{-(x-mH)} + B_1 e^{-mH} - C_1 \cdot e^x (1-2\mu_1 - mH) \cdot e^{-(x-mH)} + D_1 (1-2\mu_1 + mH) \cdot e^{-mH} \right] dm$$

If the value of H is very small (if we compute a stress near the surface), the values of $B_1 . e^{-mH}$ and $D_1 . e^{-mH}$ will converge very slowly.

We know from the preceeding paragraph that

$$B_1 e^{-mH} = 2\mu_1 e^{-mH} + A_1 (1-4\mu_1) e^{-mH} + 4\mu_1 C_1 (1-2\mu_1) e^{-mH}$$

$$D_1 e^{-mH} = e^{-mH} - 2A_1 e^{-mH} + C_1 (1-4\mu_1) e^{-mH}$$

So that

$$B_1 e^{-mH} + D_1 (1-2\mu_1 + mH) e^{-mH} = (1+mH) e^{-mH} - 2A_1 (1+mH) e^{-mH} + C_1 (1-2\mu_1 + mH - 4\mu_1 mH) e^{-mH}$$

We write then

$$\sigma_{z,H} = pa \int_0^\infty J_0(mr) \cdot J_1(ma) \left[A_1 e^x \cdot e^{-(x-mH)} - C_1 e^x (1-2\mu_1 - mH) \cdot e^{-(x-mH)} + (1+mH) e^{-mH} - 2A_1 (1+mH) e^{-mH} + C_1 (1-2\mu_1 + mH - 4\mu_1 mH) e^{-mH} \right] dm$$

We can again split this integral into

$$\sigma_{z,H} = pa \int_0^\infty J_0(mr) J_1(ma) (1+mH) e^{-mH} dm + pa \int_0^\infty J_0(mr) \cdot J_1(ma) \left[A_1 e^x e^{-(x-mH)} - C_1 e^x (1-2\mu_1 - mH) e^{-(x-mH)} - 2A_1 e^x (1+mH) \cdot e^{-(x+mH)} + C_1 e^x (1-2\mu_1 + mH - 4\mu_1 mH) e^{-(x+mH)} \right] dm$$

The first intergal can be solved analytically (§ 2.4), the second one converges normally.
Of course the same procedure can be applied for the computation of all other stresses or displacements in the first layer.

3.2.6. The stresses and displacements in the second layer.

Normally one should not have numerical problems in the computation of stresses and displacements in the second layer, except for the case that the first layer should be very thin (which can happen with overlays).
The terms $B_2 . e^{-mH}$ and $D_2 . e^{-mH}$ could then again converge quite slowly.
The solution is then obtained as follows.
We write

$$B_{21} = B_2 - \frac{1}{16(1-\mu_2)(1-\mu_3)} \cdot N_{11}(2.1, 2.2)(B_4 \quad D_4)^T$$

where $N_{11}(2.1,2.2)$ are the first and the second term of the second row of matrix N_{11}.

$$D_{21} = D_2 - \frac{1}{16(1-\mu_2)(1-\mu_3)} \cdot N_{11}(4.1, 4.2)(B_4 \quad D_4)^T$$

$$B_{41} = B_4 - \frac{(1-\mu_1)(1-\mu_2)(1-\mu_3)}{64} \left\{ \frac{1}{2}\left[F_1 k_1 L_1 (1-4\mu_1) - F_2 k_2 L_2\right] - f(xyz) \right\} \frac{1}{\nabla}$$

$$= B_4 - B_{42}$$

$$D_{41} = D_4 + \frac{(1-\mu_1)(1-\mu_2)(1-\mu_3)}{64} \cdot F_1 k_1 L_1 \cdot \frac{1}{\nabla}$$

$$= D_4 + D_{42}$$

$$N_{11}(2.1, 2.2) = (k_1 L_1 \qquad k_1 L_3 + k_3 L_2 + y(k_1 - k_2) L_2 + z(L_1 - L_2) k_1)$$

$$N_{11}(4.1, 4.2) = (0 \qquad k_2 L_2)$$

We split then $(B_4 \quad D_4)^T$ en write

$$B_2 = B_{21} + \frac{1}{16(1-\mu_2)(1-\mu_3)} N_{11}(2.1, 2.2)\left[(B_{41} \quad D_{41})^T + (B_{42} \quad D_{42})^T\right]$$

$$= B_{21} + B_{22} + \frac{1}{16(1-\mu_2)(1-\mu_3)} . N_{11}(2.1, 2.2)(B_{42} \quad D_{42})^T$$

$$D_2 = D_{21} + \frac{1}{16(1-\mu_2)(1-\mu_3)} N_{11}(4.1, 4.2)\left[(B_{41} \quad D_{41})^T + (B_{42} \quad D_{42})^T\right]$$

$$= D_{21} + D_{22} + \frac{1}{16(1-\mu_2)(1-\mu_3)} . N_{11}(4.1, 4.2)(B_{42} \quad D_{42})^T$$

We notice that the terms B_{21}, B_{22}, D_{21} and D_{22} converge normally.

$$B_{23} = \frac{1}{16(1-\mu_2)(1-\mu_3)} N_{11}(2.1, 22)(B_{42} \quad D_{42})^T =$$

$$\frac{1-\mu_1}{4}\left\{\frac{1}{2}\left[F_1 k_1 L_1 (1-4\mu_4) - F_2 k_2 L_2\right] + \hat{i}(xyz)\right\}\frac{1}{\nabla}$$

$$- \frac{1-\mu_1}{4}\left\{k_1 L_3 + L_3 L_2 + y(k_1 - k_2).L_2 + z(L_1 - L_2)k_1\right\}\frac{F_1 k_1 L_1}{\nabla}$$

$$= \frac{1-\mu_1}{4} k_1 L_1 k_2 L_2 \left\{\frac{1}{2}\left[F_1(1-4\mu_4) - F_2\right] + x(F_1 - F_2)\right\}\frac{1}{\nabla}$$

$$D_{23} = \frac{1}{16(1-\mu_2)(1-\mu_3)} N_{11}(4.1, 4.2)(B_{42} \quad D_{42})^T =$$

$$- \frac{1-\mu_1}{4} . k_2 L_2 F_1 k_1 L_1 . \frac{1}{\nabla}$$

We split now

$$\nabla = \nabla_1 + F_1 k_1 L_1 . F_2 k_2 L_2$$

where ∇_1 Converges normally.

We divide then the numerator by the denominator and obtain

$$B_{23} = \frac{1-\mu_1}{4} \cdot \frac{\frac{1}{2}\left[F_1(1-4\mu_4) - F_2\right] + x(F_1 - F_2)}{F_1 F_2}$$

$$- \frac{1-\mu_1}{4} \cdot \frac{\left\{\frac{1}{2}\left[F_1(1-4\mu_4) - F_2\right] + x(F_1 - F_2)\right\} \cdot \nabla_1}{F_1 F_2 \cdot \nabla}$$

The first term can be integrated analytically (§ 2.4), the second one converges normally.

$$D_{23} = - \frac{1-\mu_1}{4} \left[\frac{1}{F_2} - \frac{\nabla_1}{F_2 \cdot \nabla} \right]$$

3.2.7. The stresses and displacements in the third layer.

Following the same procedure as in the preceeding paragraph, we obtain

$$B_{33} = - \frac{(1-\mu_1)(1-\mu_2)}{16} \cdot \frac{\frac{1}{2}\left[F_1 K_1 (1-4\mu_3) - F_2 K_2\right] + x(F_1 - F_2) K_2 + y(K_1 - K_2) F_1}{K_1 F_1 \cdot K_2 F_2}$$

$$+ \frac{(1-\mu_1)(1-\mu_2)}{16} \cdot \frac{\left\{\frac{1}{2}\left[F_1 K_1 (1-4\mu_3) - F_2 K_2\right] + x(F_1 - F_2) K_2 + y(K_1 - K_2) F_1\right\} \nabla_1}{K_1 F_1 \cdot K_2 F_2 \cdot \nabla}$$

$$D_{33} = \frac{(1-\mu_1)(1-\mu_2)}{16} \left[\frac{1}{F_2 K_2} - \frac{\nabla_1}{F_2 K_2 \nabla} \right]$$

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APPENDIX

Three-layer elastic system

Computer program F77

```
C   PHCGRAMME    CE    TRICOUCHE   ISOTRCPE.
C
C  1ER PARTIE : DECLARATION DES VARIABLES :
C  .....................................
C
      IMPLICIT REAL(A-Z)
      INTEGER J,L,I,Q,M1,N,Z7
      CHARACTER*1 AA,BA,ZA*2
      DIMENSION J0(500),J1(500),J2(500),H(9),F(9),Y(9),II(9),JJ(9),E1(1),T(3),N7(3),N8(3),N9(3),Z1(3),Z2(3),Z3(3),G(9),
     1 ZI(2,4),ZK(4,4),ZL(4,2),ZM11(4,4),ZM12(4,4),ZM21(4,4),ZM22(4,4),ZM31(4,4),ZM32(4,4),ZM41(4,2),ZM42(4,4),
     2 ZAM(4,4),ZBM(4,2),ZCM(4,2),ZAA(2,2),ZBB(2,2),ZCC(2,2),ZDD(2,2),ZAML(4,2),ZKCM(4,2),ZM33(4,4),ZM83(4,2)
         DATA((ZI(I,J),I=1,2),J=1,4)/3*1.,-1.,0.,1.,0.,1./
         DATA((ZK(I,J),I=1,4),J=1,4)/16*0./
         DATA((ZL(I,J),I=1,4),J=1,2)/8*0./
         DATA((ZDD(I,J),I=1,2),J=1,2)/4*0./
         DATA((ZM11(I,J),I=1,4),J=1,4)/0.,0.,1.,3*0.,-1.,3*0.,1.,3*0.,-1.,0./
         DATA((ZM12(I,J),I=1,4),J=1,4)/3*0.,-1.,3*0.,-1.,3*0.,1.,3*0.,1./
         DATA((ZM21(I,J),I=1,4),J=1,4)/1.,1.,14*0./
         DATA((ZM22(I,J),I=1,4),J=1,4)/4*0.,1.,-1.,10*0./
         DATA((ZM31(I,J),I=1,4),J=1,4)/0.,0.,1.,3*0.,-1.,3*0.,1.,3*0.,-1.,0./
         DATA((ZM32(I,J),I=1,4),J=1,4)/3*0.,-1.,3*0.,-1.,3*0.,1.,3*0.,1./
         DATA((ZM41(I,J),I=1,4),J=1,2)/1.,-1.,3*0.,1.,0.,0./
         DATA((ZM42(I,J),I=1,4),J=1,2)/8*0./
C
      OPEN(1,FILE="RESMTRIISO")
C
C  2EME PARTIE : INTRODUCTION DES CONNEES :
C  .......................................
C
100   WRITE(6,2)
2     FORMAT("1DETERMINATION DES CONTRAINTES ET DEFORMATIONS ",/," DANS UNE STRUCTURE TRICOUCHE ISOTROPE ")
      WRITE(6,3)
3     FORMAT(" A SYMETRIE AXIALE.",/," ET DANS L'AXE DE LA CHARGE.",/)
      WRITE(6,998)
998   FORMAT(" ------------------------------------------------------------------------------")
      PI=3.141592
      R=0.
      ZA='AI'
      AA='N'
      BA='N'
      WRITE(6,11)
11    FORMAT(" CHOISISSEZ LE PAS D'INTEGRATION (0.1 EST EN GENERAL SUFFISANT) : ")
      READ(5,'(F5.2)')P1
      M1=P1
      M2=20.*M1
      M3=20.
104   M1=2.*M1
      M2=M2+10.*M1
      M3=M3+10.
      IF (M2.LT.256.) GOTO 104
C
C  3EME PARTIE : CALCUL DES FONCTICNS CE BESSEL.
C  ......................................................
C
      M1=P1
      M=0.
      L=20
      J=0
110   M=M+M1
      J=J+1
```

```
      IF (J.GT.M3) GOTO 120
      IF (M.GT.16.) THEN
                    J1(J)=SIN(M-PI/4.)*SQRT(2./PI/M)
              ELSE
                    Z8=M/2.
                    Z9=0.
                    V1=1.
                    V2=1.
111                 Z9=Z9+1.
                    V2=-V2/Z9*Z8**2/(Z9+1.)
                    V1=V1+V2
                    IF (ABS(V2).GT..00001) GOTO 111
                    J1(J)=Z8*V1
              END IF
      IF (J.LT.L) GOTO 110
      M1=2.*M1
      L=L+10
      GOTO 110
C
C   4EME PARTIE : INTRODUCTION DE LA STRUCTURE
C   ..........................................
C
120   WRITE(6,20)
20    FORMAT("1CARACTERISTIQUES GEOMETRIQUES DE LA STRUCTURE",//," EPAISSEUR DE LA COUCHE SUPERIEURE :")
      READ(5,'(F6.3)')H1
      WRITE(6,21)
21    FORMAT("+EPAISSEUR DE LA COUCHE INTERMEDIAIRE :")
      READ(5,'(F6.3)')H2
      WRITE(6,22)
22    FORMAT(" RAYON DE LA CHARGE  :")
      READ(5,'(F5.2)')RC
121   WRITE(6,25)
25    FORMAT(/," CARACTERISTIQUES MECANIQUES DE LA STRUCTURE")
      WRITE(6,27)
27    FORMAT(" MODULE D'ELASTICITE DE LA COUCHE SUPERIEURE ")
      READ(5,'(F12.2)')E1(1)
122   WRITE(6,35)
35    FORMAT(" MODULE D'ELASTICITE DE LA COUCHE INTERMEDIAIRE : ")
      READ(5,'(F12.2)')E2
      IF (E2.NE.E1(1)) GOTO 124
      WRITE(6,36)
36    FORMAT(" ATTENTION :... MODIFIEZ LEGEREMENT CETTE DERNIERE VALEUR")
      WRITE(6,37)
37    FORMAT(" CAR ELLE NE PEUT ETRE EGALE A UN MODULE D'ELASTICITE DE LA COUCHE SUPERIEURE.",/,"       MERCI.")
      GOTO 122
124   WRITE(6,40)
40    FORMAT(" MODULE ELASTIQUE DU COFFRE : ")
      READ(5,'(F12.2)')E3
      IF (E3.NE.E2) GOTO 125
      WRITE(6,41)
41    FORMAT(" ATTENTION.. MODIFIEZ LEGEREMENT CE COEFFICIENT CAR IL DOIT ETRE DIFFERENT DE CELUI DE LA COUCHE INTERMEDIAIRE ")
      GOTO 124
125   WRITE(6,44)
44    FORMAT("+PRESSION APPLIQUEE EN SURFACE : ")
      READ(5,'(F5.2)')P
C
C  5EME PARTIE : EXECUTION DES CALCULS.
C  ..................................
C
         Q=1
```

```
201       H1=0
          P0=P1
          K=E1(Q)/E2
          T9=E2/E3
          ZK(1,1)=1.+K
          ZK(2,2)=1.+K
          ZK(3,3)=1.+K
          ZK(4,4)=1.+K
          ZL(2,1)=1.+T9
          ZL(4,2)=1.+T9
          ZDD(1,1)=(1.+K)*(1.+T9)
          ZDD(2,1)=-(1.+K)*(1.+T9)
          ZDD(2,2)=(1.+K)*(1.+T9)
          M=P0
          J=1
          L9=20.*P0
          L=20
          DO 211 I=1,9
             Y(I)=0.
             JJ(I)=0.
211       CONTINUE
C
C         CALCUL DE JJ(1) A L'ORIGINE.
C
          Z8=1.
          Z9=(1.-K*T9)*(-4.*T9+6.*K+2.*K*T9)
          Z9=Z9/(-4.*T9+6.*K+3.*K*K-2*K*K*T9+2*K*T9*T9-K*K*T9*T9)
          Z9=Z9*P0/3
          JJ(1)=(Z8-Z9)/E1(Q)
C
C         FIN DU CALCUL DE JJ(1) A L'ORIGINE
C
C
C         CALCUL DES CONSTANTES D'INTEGRATION
C
214       X=M*H1/RC
          Y9=M*(H1+H2)/RC
          X1=EXP(-X)
          X2=X1*X1
          X3=X1*X2
          X4=X2*X2
          Y1=EXP(-Y9)
          Y2=Y1*Y1
          Y3=Y1*Y2
          Y4=Y2*Y2
          XY=2*(Y9-X)
          X5=EXP(-XY)
          XY1=2*Y9-X
          Y5=EXP(-XY1)
          XY2=Y9-X
          X6=EXP(-XY2)
          ZM11(1,1)=-1.-X
          ZM11(1,2)=X
          ZM11(1,3)=-1.-X
          ZM11(1,4)=X
          ZM12(2,1)=-1.+X
          ZM12(2,2)=X
          ZM12(2,3)=1.-X
          ZM12(2,4)=-X
          ZM21(3,1)=K
```

```
      ZM21(4,1)=K
      ZM21(1,3)=X
      ZM21(2,3)=1.+X
      ZM21(3,3)=K*X
      ZM21(4,3)=K*(1.+X)
      ZM22(3,2)=-K
      ZM22(4,2)=K
      ZM22(1,4)=X
      ZM22(2,4)=1.-X
      ZM22(3,4)=-K*X
      ZM22(4,4)=-K*(1.-X)
      ZM31(1,1)=-1.-Y9
      ZM31(1,2)=Y9
      ZM31(1,3)=-1.-Y9
      ZM31(1,4)=Y9
      ZM32(2,1)=-1+Y9
      ZM32(2,2)=Y9
      ZM32(2,3)=1.-Y9
      ZM32(2,4)=-Y9
      ZM41(3,1)=-T9
      ZM41(4,1)=T9
      ZM41(4,2)=-T9
      ZM42(1,2)=Y9
      ZM42(2,2)=-Y9
      ZM42(3,2)=-T9*Y9
      ZM42(4,2)=T9*Y9
      CALL PRODUIT4444(ZM11,ZM22,ZAM)
      CALL SOMME42(ZM41,ZM42,ZM83)
      CALL PRODUIT4442(ZM31,ZM83,ZCM)
      CALL PRODUIT4444(ZM12,ZM21,ZM33)
      CALL PRODUIT4442(ZM33,ZCM,ZBM)
      CALL PRODUIT4442(ZAM,ZL,ZAML)
      CALL PRODUIT4442(ZK,ZCM,ZKCM)
      CALL PRODUIT2442(ZI,ZAML,ZAA)
      CALL PRODUIT2442(ZI,ZBM,ZBB)
      CALL PRODUIT2442(ZI,ZKCM,ZCC)
      DET1=(-ZAA(1,1)*X2+ZBB(1,1)*X5-ZCC(1,1)*Y2)*(-ZAA(2,2)*X2+ZBB(2,2)*X5-ZCC(2,2)*Y2)
      DET1=DET1-(-ZAA(1,2)*X2+ZBB(1,2)*X5-ZCC(1,2)*Y2)*(-ZAA(2,1)*X2+ZBB(2,1)*X5-ZCC(2,1)*Y2)
      DET1=DET1+ZDD(1,1)*(-ZAA(2,2)*X2+ZBB(2,2)*X5-ZCC(2,2)*Y2)
      DET1=DET1+ZDD(2,2)*(-ZAA(1,1)*X2+ZBB(1,1)*X5-ZCC(1,1)*Y2)
      DET1=DET1-ZDD(2,1)*(-ZAA(1,2)*X2+ZBB(1,2)*X5-ZCC(1,2)*Y2)
      DET2=ZDD(1,1)*ZDD(2,2)
      DET=DET1+DET2
      DETB3=4.*(-ZAA(2,2)*X2+ZBB(2,2)*X5-ZCC(2,2)*Y2+ZDD(2,2))/DET
      DETD3=-4.*(-ZAA(2,1)*X2+ZBB(2,1)*X5-ZCC(2,1)*Y2+ZDD(2,1))/DET
      DETA2=-.5*Y1*(ZCM(1,1)*DETB3+ZCM(1,2)*DETD3)
      DETB2=.5*(1.+T9)*DETB3
      DETC2=-.5*Y1*(ZCM(3,1)*DETB3+ZCM(3,2)*DETD3)
      DETD2=.5*(1.+T9)*DETD3
      DETA1=.25*(-X1*(ZAML(1,1)*DETB3+ZAML(1,2)*DETD3)-Y5*(ZKCM(1,1)*DETB3+ZKCM(1,2)*DETD3))
      DETC1=.25*(-X1*(ZAML(3,1)*DETB3+ZAML(3,2)*DETD3)-Y5*(ZKCM(3,1)*DETB3+ZKCM(3,2)*DETD3))
      DETB1=.25*(X5*(ZBM(2,1)*DETB3+ZBM(2,2)*DETD3)+(1.+T9)*(1.+K)*DETB3)
      DETD1=.25*(X5*(ZBM(4,1)*DETB3+ZBM(4,2)*DETD3)+(1.+T9)*(1.+K)*DETD3)
C
C     FIN DU CALCUL DES CONSTANTES D'INTEGRATION
C
      H(1)=-2.*DETA1*X1/M/E1(Q)
      H(2)=DETA1+DETB1*X1+DETC1*X+DETD1*X*X1
      H(3)=-.5*(DETA1+DETB1*X1+(3.+X)*DETC1-(3.-X)*DETD1*X1)
      H(4)=DETA2*X6+DETB2*X1+DETC2*X*X6+DETD2*X*X1
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      M(5)=-.5*(DETA2*X6+DETB2*X1+(3.+X)*DETC2*X6-(3.-X)*DETD2*X1)
      M(6)=DETA2+DETB2*Y1+DETC2*Y9+DETD2*Y9*Y1
      M(7)=-.5*(DETA2+DETB2*Y1+(3.+Y9)*DETC2-(3.-Y9)*DETD2*Y1)
      M(8)=DETB3*Y1+Y9*DETD3*Y1
      M(9)=-.5*(DETB3*Y1-DETD3*(3.-Y9)*Y1)
      DO 230 I=1,9
         F(I)=M(I)*J1(J)
230   CONTINUE
      IF (W1.EQ.0) THEN
                        DO 235 I=1,9
                           F(I)=4.*F(I)
235                     CONTINUE
                        W1=1
                        M=M+P0
                        J=J+1
                  ELSE
                        DO 236 I=1,9
                           F(I)=2.*F(I)
236                     CONTINUE
                        W1=0
                  END IF
      DO 237 I=1,9
         Y(I)=Y(I)+F(I)
237   CONTINUE
      Z7=1
      IF (W1.EQ.1) GOTO 214
      E=.000001
      DO 238 I=1,9
         IF(ABS(M(I)).GT.E) GOTC 239
238   CONTINUE
      GOTO 245
239   IF (J.GE.L) THEN
                       DO 240 I=1,9
                          Y(I)=Y(I)+F(I)/2.
                          II(I)=P0*Y(I)/3.
                          JJ(I)=JJ(I)+II(I)
                          Y(I)=0.
240                    CONTINUE
                       P0=2.*P0
                       L=L+10

                       L9=2.*L9
                 END IF
      M=M+P0
      J=J+1
      Z7=1
      GOTO 214
245   DO 246 I=1,9
         II(I)=P0*Y(I)/3.
         JJ(I)=JJ(I)+II(I)
246   CONTINUE
      DO 895 I=1,9
         G(I)=P*JJ(I)
895   CONTINUE
      G(1)=G(1)*1.5*RC
C
C     6EME PARTIE : IMPRESSION DES CONTRAINTES ET DES DEPLACEMENTS.
C     .................................................................
C
      WRITE(1,'(///)')
```

```
      WRITE(6,70)
70    FORMAT("1",//)
      WRITE(1,301)
301   FORMAT(" RESULTATS DANS L'AXE DE LA CHARGE.",/)
      WRITE(1,71)E1(Q),H1,RC,P
      WRITE(6,71)E1(Q),H1,RC,P
71    FORMAT("  E1=",F7.0,"    U1=0.50         H1=",F5.2,"     A=",F6.2,"      P=",F5.2)
      WRITE(1,72)E2,H2,R
      WRITE(6,72)E2,H2,R
72    FORMAT("  E2=",F7.0,"    U2=0.50         H2=",F5.2,"     R=",F6.2)
      WRITE(1,73)E3
      WRITE(6,73)E3
73    FORMAT("  E3=",F7.0,"    U3=0.50 ",//)
      WRITE(1,74)
      WRITE(6,74)
74    FORMAT("                         SURFACE    BASE        SURFACE     BASE       SURFACE")
      WRITE(1,75)
      WRITE(6,75)
75    FORMAT("                                    COUCHE1     COUCHE2     COUCHE2    COUCHE3",/)
      WRITE(1,76)G(2),G(4),G(6),G(8)
      WRITE(6,76)G(2),G(4),G(6),G(8)
76    FORMAT(" CONTRAINTES VERTICALES   *    ",F10.6,"  ",F10.6,"  ",F10.6,"  ",F10.6)
      WRITE(1,77)G(3),G(5),G(7),G(9)
      WRITE(6,77)G(3),G(5),G(7),G(9)
77    FORMAT(" CONTRAINTES RADIALES     *    ",F10.6,"  ",F10.6,"  ",F10.6,"  ",F10.6)
      WRITE(1,78)G(3),G(5),G(7),G(9)
      WRITE(6,78)G(3),G(5),G(7),G(9)
78    FORMAT(" CONTRAINTES CIRCONF.     *    ",F10.6,"  ",F10.6,"  ",F10.6,"  ",F10.6)
      WRITE(1,79)G(1)
      WRITE(6,79)G(1)
79    FORMAT(" FLECHES              ",F7.4,"       *           *           *          *")
C
C        FIN D'IMPRESSION.
C        .................
C
     WRITE(6,'(" ",//," LES RESULTATS SONT SUR LE FICHIER  RESMTRIISO."
    1 ,//,"          C''EST TERMINE !!!",//,"               MERCI.")')
     END
C
     SUBROUTINE PRODLIT4444(A,B,C)
     DIMENSION A(4,4),B(4,4),C(4,4)
     DO 800 I=1,4
       DO 801 K=1,4
         C(I,K)=0.
         DO 802 J=1,4
            C(I,K)=C(I,K)+A(I,J)*B(J,K)
802      CONTINUE
801    CONTINUE
800  CONTINUE
     END
C
     SUBROUTINE PRODLIT2442(A,B,C)
     DIMENSION A(2,4),B(4,2),C(2,2)
     DO 810 I=1,2
       DO 811 K=1,2
         C(I,K)=0.
         DO 812 J=1,4
            C(I,K)=C(I,K)+A(I,J)*B(J,K)
812      CONTINUE
811    CONTINUE
```

```
810     CONTINUE
        END
C
        SUBROUTINE PRODLIT4442(A,B,C)
        DIMENSION A(4,4),B(4,2),C(4,2)
        DO 830 I=1,4
          DO 831 K=1,2
            C(I,K)=0.
            DO 832 J=1,4
               C(I,K)=C(I,K)+A(I,J)*B(J,K)
832         CONTINUE
831       CONTINUE
830     CONTINUE
        END
C
        SUBROUTINE SOMME42(A,B,C)
        DIMENSION A(4,2),B(4,2),C(4,2)
        DO 860 I=1,4
           DO 861 J=1,2
              C(I,J)=A(I,J)+B(I,J)
861        CONTINUE
860     CONTINUE
        END
```

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